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PRECISE COMPUTATION BY NUMERICAL INTEGRATION OF VERY LONG GEODESIC AND COMPARISON WITH APPROXIMATE METHODS

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FOREWORD

In the UN Convention of the Law of the Sea, several articles prescribe the outer limit lines, lines of delimitation and median lines. These lines should be drawn according to the results of precise geodetic computation. The distances on the Earth's surface are computed according to the geodesic length on an ellipsoid surface.

For computation of the geodesic, several methods are known. Some of them are based on series expansion. Some of them are derived from spherical trigonometry. The method described here is based on numerical integration. It is directly derived from the EULER's equation of the calculus of variations.

This method can be applied to a very long distance, more than 19,000km, with an accuracy of 1mm. Algorithms and some examples are described here.

1. EULER'S EQUATION OF THE CALCULUS OF VARIATIONS

A line element ds on an ellipsoid surface is formulated as follows:

$$\begin{aligned} ds^2 &= dx^2 + dy^2 + dz^2 \\ &= r_m^2\, d\phi^2 + r_p^2\, d\lambda^2 = (r_m^2 + r_p^2\dot{\lambda}^2)\, d\phi^2 \qquad \left(\dot{\lambda} = \frac{d\lambda}{d\phi}\right) \end{aligned}$$

where r_m and r_p are the radius of curvature of the meridian and radius of curvature of the parallel of latitude, respectively, and described as follows:

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$$r_m = a(1-e^2)/w^{3/2}$$
$$r_p = a \cos\phi/w = N \cos\phi$$
$$(w = \sqrt{1-e^2 \sin^2\phi})$$

where N is the radius of curvature of the prime vertical, ϕ and λ are latitude and longitude respectively.

The line elements ds becomes as follows, by using σ_c as the sign of change.

$$ds = \sigma_c \sqrt{r_m^2 + r_p^2 \dot{\lambda}^2}\, d\phi$$
$$\sigma_c = +1 \text{ (when } \phi \text{ is increasing, i.e. } \cos\alpha>0)$$
$$= -1 \text{ (when } \phi \text{ is increasing, i.e. } \cos\alpha<0)$$

According to the general theory of the calculus of variations, the stationary function y, which gives a stationary value to the following definite integration L:

$$L = \int_{x_1}^{x_2} F(x,y,\dot{y})dx$$

is decided by the following EULER's equation:

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial \dot{y}}\right) = 0$$

By the application of this theory to the present problem, the following formulae are obtained.

$$\sigma_c \frac{r_p^2 \dot{\lambda}}{\sqrt{r_m^2 + r_p^2 \dot{\lambda}^2}} = K$$
$$\dot{\lambda} = \sigma_c K \frac{r_m}{r_p\sqrt{r_p^2 - K^2}}$$

By substitution, the line element ds is obtained as follows.

$$ds = \sigma_c \frac{r_m r_p}{\sqrt{r_p^2 - K^2}} d\phi$$

Introduction of the azimuth α brings the equation to the following well known equation of the geodesic.

$$\tan \alpha = \frac{r_p}{r_m} \dot{\lambda}$$

$$r_p \sin \alpha = K$$

The geodesic is determined by the parameter K, that is established by the two terminal points of the geodesic. There are two kinds of terminal dispositions of the geodesic. On the one hand, two terminal points, P_1 and P_2, lie on the same side of the ascending course or the descending course, as shown in Fig. 1. On the other hand, two terminal points lie on different sides of the course, as shown in Fig. 2. The top point P_x, where the latitude takes the minimum value in the course, lies outside of the course in the former case, and inside the course in the latter case.

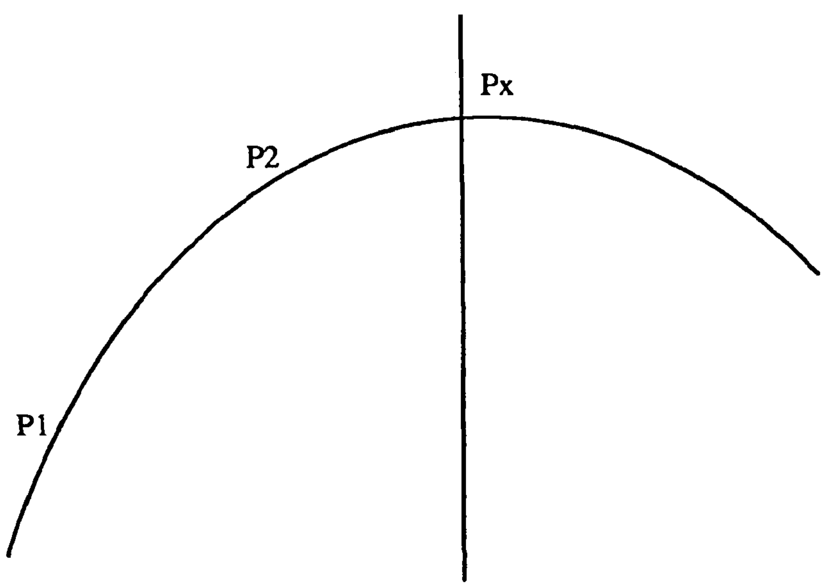

FIG. 1.- Single route.
P1 and P2 lie in the same ascending course.

In this report, the former case is called the "Single Route", and the latter case is called the "Turn Route". The turn route appears not only in the case of a long distance, but also in the case of a short distance, such as that of equal latitudes of two terminal points.

The formulae of the integral are described by using the latitudes and longitudes (ϕ_1, λ_1), (ϕ_2, λ_2) and (ϕ_x, λ_x) of the terminals P_1, P_2 and the top point P_x.

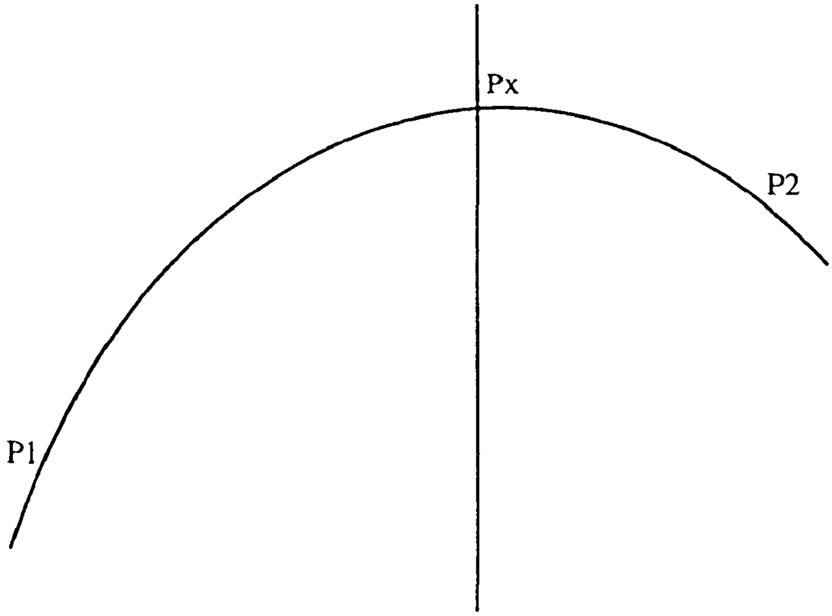

FIG. 2.- Turn route.
P1 and P2 lie on different sides of the course.

1. Single Route

In this case, the latitude simply increases or decreases from ϕ_1 to ϕ_2. The following formulae are obtained.

$$\lambda_2' = \sigma_c \, K \int_{\phi_1}^{\phi_2} \frac{r_m}{r_p \sqrt{r_p^2 - K^2}} \, d\phi + \lambda_1$$

$$s = \sigma_c \int_{\phi_1}^{\phi_2} \frac{r_m \, r_p}{\sqrt{r_p^2 - K^2}} \, d\phi$$

Partial derivatives of λ'_2 and s with respect to K are necessary for reiteration and they can easily be obtained. The sign coefficient σ_c is constant through the whole integral course. It can be obtained at point P_1. Reiteration can be done by using the following ΔK, modification of the parameter K, where K_o is the original value of K.

$$\Delta K = \frac{\Delta\lambda}{I_1 + K_0^2 I_2}$$

$$\Delta\lambda = \lambda_2 - \lambda_2'$$

$$I_1 = \sigma_c \int_{\phi_1}^{\phi_2} \frac{r_m}{r_p\sqrt{r_p^2 - K_0^2}}\, d\phi$$

$$I_2 = \sigma_c \int_{\phi_1}^{\phi_2} \frac{r_m}{r_p(r_p^2 - K_0^2)^{3/2}}\, d\phi$$

2. Turn Route

In this case, the integrals of λ_2 and s are similarly determined by the coordinates of P_1 and P_2. However, the integrals are composed of two terms, one integral from ϕ_1 to ϕ_x, and the other from ϕ_2 to ϕ_x. The latitude ϕ_x of the top point P_x is determined by the following formula.

$$\cos\phi_x = |K|\, \frac{\sqrt{1-e^2}}{\sqrt{a^2 - e^2K^2}}$$

The sign of the latitude ϕ_x, is similar to that of cos α at P_1.

By the use of ϕ_x, K and σ_c at P_1, the longitudes λ_x, λ_2 and distance s are computed by the following formulae.

$$\lambda_x = \lambda_1 + \sigma_c K \int_{\phi_1}^{\phi_x} \frac{r_m}{r_p\sqrt{r_p^2 - K^2}}\, d\phi = \lambda_2 - \sigma_c K \int_{\phi_2}^{\phi_x} \frac{r_m}{r_p\sqrt{r_p^2 - K^2}}\, d\phi$$

$$\lambda_2 = \lambda_1 + \sigma_c K \int_{\phi_1}^{\phi_x} \frac{r_m}{r_p\sqrt{r_p^2 - K^2}}\, d\phi + \sigma_c K \int_{\phi_2}^{\phi_x} \frac{r_m}{r_p\sqrt{r_p^2 - K^2}}\, d\phi$$

$$s = \sigma_c \int_{\phi_1}^{\phi_x} \frac{r_m r_p}{\sqrt{r_p^2 - K^2}}\, d\phi + \sigma_c \int_{\phi_2}^{\phi_x} \frac{r_m r_p}{\sqrt{r_p^2 - K^2}}\, d\phi$$

For the turn route, when K varies, not only the integral function but also the top of the integral area ϕ_x varies according to the above stated function. For reiteration, the modification ΔK of K is computed by the use of successive sets of approximation of (K_1, λ_2') and (K_2, λ_2'').

$$\Delta K = \frac{K_2 - K_1}{\Delta\lambda_2 - \Delta\lambda_1} (-\Delta\lambda_1)$$

$$\begin{aligned} \Delta\lambda_1 &= \lambda_2 - \lambda_2' \\ \Delta\lambda_2 &= \lambda_2 - \lambda_2'' \\ K &= K_1 - \sigma_\lambda |K| \\ \sigma_\lambda &= +1 \ (\textit{if } \Delta\lambda_1 > 0) \\ &= -1 \ (\textit{if } \Delta\lambda_1 < 0) \end{aligned}$$

2. NUMERICAL INTEGRATION

The above stated formulae of λ_x, λ_2 and s are written in integral form. In this report, they are directly computed by numerical integration, not by series expansion. There are two ways of numerical integration applicable to the present problem. One is the method of the weighted mean, and the other is the method of the double hyperbolic function transformation. The former is applicable to the integration of the single route. The latter is applicable not only to the integration of the single route but also to that of the turn route, in which the upper terminal of the integration area is a singular point of the integration function.

1. Weighted Mean Method

In this method, the value of the following integration I is approximated by the following weighted mean S.

$$I = \int_{-1}^{+1} f(x)\,dx$$
$$S = \Sigma\, w_i f(x_i)$$

There are many methods to calculate the weighted mean, such as the methods of NEWTON-COTE, MACLAURIN and GAUSS. In GAUSS's method, the values of x_i and w_i are determined so as to get the best approximation. The GAUSS's method of six division points is adopted. The division of the area x_i and the weight values w_i are as follows:

-x1=x6=0.932469514203152
-x2=x5=0.661209386466265
-x3=x4=0.238619186083197

w1=w6=0.171324492379170
w2=w5=0.360761573048139
w3=w4=0.467913934572691

The integration is repeated until the integrated value becomes sufficiently converged. The limit ε of reiteration convergence can be set to 1mm or less in value.

2. Double Hyperbolic Function Method

For the turn route, the upper terminal of the integral area is a singular point of the integral function. In this case, the above stated GAUSS's method becomes very slow in converging. Another method is suggested. The method of the double hyperbolic function is suitable. Here, the singularity of p, q>-1 at both terminals +1 of the following integration function can be allowed.

$$I = \int_{-1}^{+1} f(x)\,dx$$

By applying the transformation

$$x=g(t)$$

the integral area (-1,+1) is transformed to $(-\infty,+\infty)$.

$$I = \int_{-\infty}^{+\infty} f(g(t))\, g'(t)\, dt$$

This integral resolves itself into the following summation by dividing the area into small divisions with step h.

$$I = h \sum_{-\infty}^{+\infty} f\,(g(nh))\; g'(nh)$$

As the form of the transformation g(t), functions that decrease to zero in a double hyperbolic function manner when $|t|$ goes to infinity, are known to be most suitable. The following function meets this condition and is used in the computation of this report.

$$x = \tanh\left(\frac{\pi}{2} \sinh(t)\right)$$

As for infinite summation, it is terminated when the value of the additional term becomes sufficiently small and no significant change occurs by omission of the succeeding terms. The reiteration is terminated by the convergence limit ε.

3. FIRST APPROXIMATION

For reiteration, the first approximation of the geodetic parameter K is necessary. The approximate value of K is computed from the approximate value of the azimuth α.

1. Approximate azimuth at point P_1

As an approximation of azimuth α, the following angle α' can be computed by using the position vectors $\vec{r}_1$, $\vec{r}_2$ of P_1 and P_2, and the north unit vector $\vec{N}$ and the east unit vector $\vec{E}$ at P_1.

$$\tan \alpha' = \frac{(\vec{r}_2 - \vec{r}_1) \cdot \vec{E}}{(\vec{r}_2 - \vec{r}_1) \cdot \vec{N}}$$

Another approximation α'' can be computed by the following formula:

$$\tan \alpha'' = - \frac{((\vec{r}_2 - \vec{r}_1) \times \vec{n}_2) \cdot \vec{N}}{((\vec{r}_2 - \vec{r}_1) \times \vec{n}_2) \cdot \vec{E}}$$

where $\vec{n}_2$ is the normal unit vector at point P_2.

The mean α_1 of α' and α'' has an accuracy of about 0".01 at distance of about 130 km, 0".1 at 530 km and 0".8 at 1320 km in comparison with the standard examples shown in reference 1.

2. Approximate distance by mid-latitude azimuth

The following formula of cord length l, radius ρ and arc length s can be used for approximation of distance.

$$s = 2 \sin^{-1} \left(\frac{l}{2\rho}\right)$$

where l is computed by the following formula.

$$l = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

The radius of curvature ρ is computed by the azimuth α_m at mid-latitude ϕ_m.

$$\sin \alpha_m = \sin \alpha_1 \frac{r_p(\phi_1)}{r_p(\phi_m)}$$

$$\frac{l}{\rho} = \frac{\cos^2\alpha_m}{r_m(\phi_m)} + \frac{\sin^2\alpha_m}{N(\phi_m)}$$

The accuracy of this approximation is very good, 0 mm at 130 km, 2 mm at 530 km and 6 cm at 1320 km in comparison with the standard examples shown in reference 1.

3. Approximate distance by scalar product

For the turn route, another formula gives a better approximation, which is based on the vector angle computed by the scalar product of two vectors. First, a point Q is taken on the z-axis apart from the origin by z_q, and computed by the following formula using the mid-latitude ϕ_m.

$$z_q = -\frac{a\, e^2 \sin \phi_m}{\sqrt{1-e^2\sin^2\phi_m}}$$

Then the angle θ between the vectors $\overrightarrow{QP_1}$ and $\overrightarrow{QP_2}$ is computed from the scalar product formula.

$$\cos \theta = \frac{\vec{QP}_1 \cdot \vec{QP}_2}{|\vec{QP}_1| \cdot |\vec{QP}_2|}$$

From this angle θ, the radius ρ and the distance s are computed by using the cord length l.

$$\rho = \frac{l}{2\sin(\theta/2)}$$

$$s = \theta\rho$$

The accuracy of this approximation in the case of the turn route is very good, 0 mm at about 370 km, 2 mm at 920 km and 1 mm at 1200 km in comparison with the results of the integration method as shown in the examples stated later.

4. DISCRIMINATION OF INTEGRATION ROUTE

1. Coordinate transformation

To choose the integration method, it is necessary to discriminate the integration route. Three points P_1, P_2 and the origin O determine a plane. The cut of this plane and the ellipsoidal surface is proved to be an ellipse. The equatorial semi-axis of this ellipse is identical to that of the ellipsoid. The polar semi axis b′ is computed by the following formula:

$$\frac{1}{b'^2} = \frac{\cos^2 i}{a^2} + \frac{\sin^2 i}{b^2}$$

where i is the inclination of the plane containing the ellipse to the equatorial plane, as shown in Fig. 3. The inclination angle i and the rotation angle Ω are computed from the coordinates (x_1, y_1, z_1), (x, y_2, z_2) of points P_1 and P_2 through the direction cosine 1,m,n.

$$i = \tan^{-1}\left(\frac{\sqrt{1^2+m^2}}{n}\right)$$

$$\Omega = \tan^{-1}\left(\frac{1}{-m}\right)$$

$$\begin{pmatrix} l \\ m \\ n \end{pmatrix} = \begin{pmatrix} \sin i \sin \Omega \\ -\sin i \cos \Omega \\ \cos i \end{pmatrix}$$

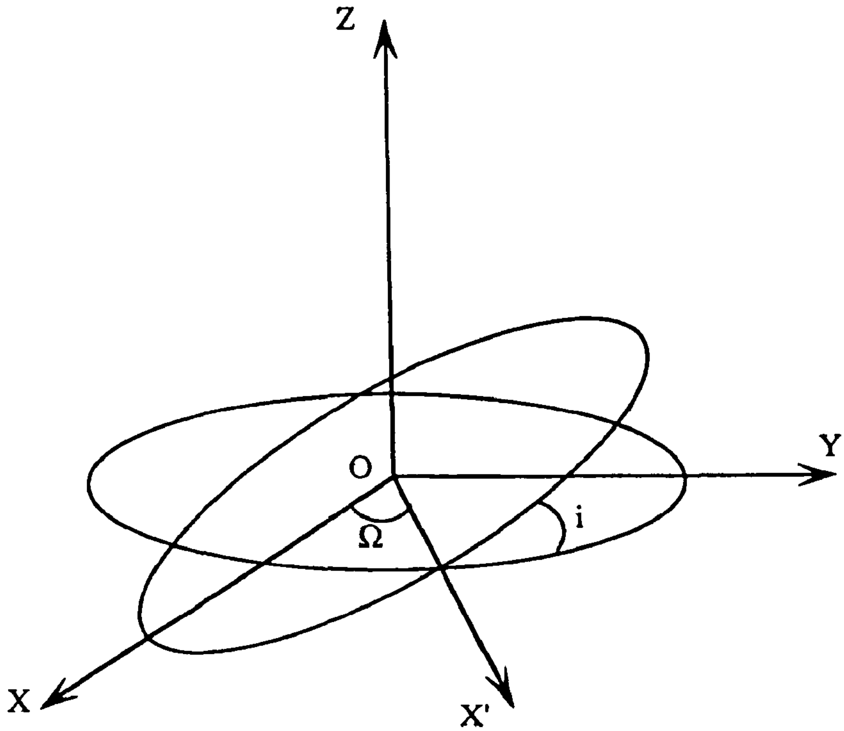

FIG. 3.- Coordinate transformation.
Ω: Rotation around z-axis; i: Rotation around x'-axis.

The direction cosine l,m,n are computed by the expansion of the following determinant.

$$\begin{pmatrix} l \\ m \\ n \end{pmatrix} = A \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

where A is the normalization factor and $\vec{i}$, $\vec{j}$, $\vec{k}$ are unit vectors in the direction of the x-axis, y-axis and z-axis respectively.

The new vector r' in the cut plane is transformed from the original vector r by the following rotation matrices Rx(i) and Rz(Ω).

$$r' = Rx(i)\ Rz(\Omega)r$$

where matrix Rx(i) denotes rotation around the x-axis by angle i, and Rz(Ω) around the z-axis by Ω. After this rotation, if the transformed coordinates y'_1 and y'_2 of the points P_1 and P_2 have the same sign, the integration route is the single route. If, on the other hand, they have different sign, the route is the turn route. This is the discrimination of the route.

2. Approximate Distance by Inclined Ellipse

As a byproduct of the coordinate transformation, there arises another approximation of distance. The angle ψ of the normal to the ellipse is computed by the following formula, where e' is the eccentricity of the ellipse.

$$\psi = \tan^{-1} \left(\frac{y'}{(1-e'^2)x'}\right)$$

From the mean angle ψ_m of the two normal angles ψ_1 and Ψ_2 at points P_1 and P_2, the radius ρ of the curvature of the ellipse can be computed.

$$\rho = \frac{a(1-e'^2)}{(1-e'^2\sin^2 \psi_m)^{3/2}}$$

By substituting this radius ρ in the formula of cord and arc, an approximate distance can be computed. This gives a better approximation than the approximate distance computed by the radius of curvature from azimuth α_m at mid-latitude ϕ_m. The difference is 0 mm at 130 km and 530 km, and 1.8 cm at 1320 km in comparison with the standard examples shown in reference 1.

5. EXAMPLES

Several examples are computed for both single and turn route.

1. Single Route

Five examples, cases A through E, are shown in Table 1. The results computed by two methods of integration are shown as dist.gm and dist.dh, which are distances computed by weighted mean using GAUSS's method and the double hyperbolic function method, respectively.

Approximate distances are also shown as dist.ma and dist.ie which are computed from mid-latitude azimuth and included ellipse respectively.

For the convenient comparison, the input data of the cases A through C are the same as those of the standard examples shown in reference 1. They are said to be originally the examples shown in the Handbuch der Vermessungskunde by JORDAN-EGGERT-KNEISSL (1959). The input data of case D are also similar to those of the example in reference 2. Those results coincide very well with the former computations.

The case E is a completely original example and has no former results for comparison. The approximate distance is computed using vector method. In this case, because of very long distances, dist.ma and dist.ie are not so good approximations as in the other cases.

Case	A	B	C	D	E
Input Data					
$\phi 1$	49° 30′ 0″	52° 30′ 16.7″	45° 0′ 0″	10° 0′ 0″	35° 0′ 0″
$\lambda 1$	0° 0′ 0″	0° 0′ 0″	0° 0′ 0″	0° 0′ 0″	140° 0′ 0″
$\phi 2$	50° 30′ 0″	54° 42′ 50.6″	55° 0′ 0″	55° 0′ 0″	-35° 0′ 0″
$\lambda 2$	1° 0′ 0″	7° 6′ 0″	10° 0′ 0″	49° 35′ 55.480210″	316° 0′ 0″
Approximation					
dist.ma(m)	132315.375	529979.580	1320284.430	6606991.305	19650401.991*
dist.ie(m)	132315.375	529979.578	1320284.347	6606733.567	
Results					
dist.gm(m)	132315.375	529979.578	1320284.368	6606996.043	19661372.255
dist.dh(m)	132315.375	529979.578	1320284.368	6606996.043	19661372.255

Table 1. Examples of single route.

dist.ma : Approximate distance by mid-latitude azimuth
dist.ie : Approximate distance by inclined ellipse
* : Approximate distance by vector angle
dist.gm : Integrated distance by GAUSS's weighted mean
dist.dh : Integrated distance by double hyperbolic function

Ellipsoid : Bessel 1841 (cases A to C) a=6377397.155m f=1/299.152813
International(case D) a=6378388.m f=1/297.
WGS 84(case E) a=6378137.m f=1/298.257223563

Convergence limit: ε =0.1mm

2. Turn Route

Five examples are shown in Table 2. The cases A through C are computed as examples of short and medium distances. The case D and E are examples of long and very long distances.

The results are computed by the double hyperbolic function method and shown as dist.dh. Approximate distances are computed by the mid-latitude azimuth method, inclined ellipse and vector angle by scalar product, represented as dist.ma, dist.ie and dist.va, respectively. For the cases D and E, dist.ma and dist.ie are not so good approximations and result in big errors in comparison with other cases.

In the case of the turn route, there are no examples which were previously computed or can be easily found, so no comparison is made with these examples.

Case	A	B	C	D	E
Input Data					
$\phi 1$	34° 0′ 0″	34° 0′ 0″	34° 0′ 0″	31° 52′ 42″	1° 0′ 0″
$\lambda 1$	131° 0′ 0″	130° 0′ 0″	130° 0′ 0″	130° 54′ 15″	1° 0′ 0″
$\phi 1$	34° 0′ 0″	34° 0′ 0″	34° 0′ 0″	32° 3′ 20″	1° 0′ 0″
$\lambda 1$	135° 0′ 0″	140° 0′ 0″	143° 0′ 0″	35° 17′ 29″	175° 0′ 0″
Approximation					
dist.ma(m)	369471.650	923370.475	1200051.050	8680101.745	19498109.734
dist.ie(m)	369471.650	923370.456	1200050.968	8677078.970	19267006.900
dist.va(m)	369471.650	923370.457	1200050.984	8678076.042	19333513.953
Results					
dist.dh(m)	369471.650	923370.459	1200050.983	8677723.647	19330340.907

Table 2. Examples of turn route.

dist.ma : Approximate distance by mid-latitude azimuth
dist.ie : Approximate distance by inclined ellipse
dist.va : Approximate distance by vector angle
dist.dh : Integrated distance by double hyperbolic function

Ellipsoid : Bessel 1841 (cases A to C) a=6377397.155m f=1/299.152813
WGS 84(cases D to E) a=6378137.m f=1/298.257223563
Convergence limit: ε =0.1mm (cases A to C), ε =1mm (cases D to E)

6. CONCLUSIONS

Two kinds of precise computation of very long distance geodesic by numerical integration, GAUSS's weighted mean method and the double hyperbolic function method, were developed for single route. They all agree together, with an accuracy of 1 mm over distances of more than 19,000 km.

One method, the double hyperbolic function method, can be applied to the turn route, over a distance of more than 19,000 km to an accuracy of 1 mm.

Three kinds of approximate computation for long distance geodesic were developed by using geometry and vectors. They are the methods of the mid-latitude azimuth, the included ellipse and the vector angle. The first two methods are applicable to both of single and turn routes. The vector angle method has good accuracy for the turn route.

These three approximate methods can be used up to 1,200km for practical use with good accuracy. At 200 nautical miles, they have an accuracy of 1mm.

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