

TOWARDS A REAL-TIME TIDAL ANALYSIS AND PREDICTION

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Abstract

In the practice of tidal analysis and prediction, the number and kind of astronomical tidal components that are to be included in a tidal model depend on the length of available tidal record and the desired accuracy of prediction. Since tidal frequencies, including shallow water constituents, are distributed unequally in a few narrow frequency bands, an inappropriate selection of tidal constituents to be included in the analysis and prediction may cause the normal equations to become ill-conditioned, or even singular, and the prediction to become poor. This investigation shows how to construct lumped tidal frequencies which better characterize ocean tides with diminishing length of observational series.

Further, a sequential tidal analysis model is proposed and an algorithm for its implementation is presented, which can rigorously update a tidal solution when the number of observations increases. The algorithm also brings in automatically additional tidal constituents without a large amount of computation work; the CPU time for this analysis is only about 4 percent of that for the conventional harmonic technique. The sequential algorithm for ocean tidal analysis and prediction has a potential to be used in tide gauge stations for providing continuous up-to-date tidal prediction.

INTRODUCTION

The drive towards an increasingly more accurate predictive capability of sea level elevation in coastal zones has been spurred on by concerns relating to navigation, global warming, shore-line engineering and pollutant transport. Traditionally, site specific sea level tidal information is derived from the harmonic analysis of a collected time series which estimates the amplitudes and phases of

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some selected harmonic tidal constituents. Then, these harmonic tidal constituents, which can be either of an astronomic origin, or of the shallow water variety [ZETLER and ROBERT, 1967], are used to predict the tide for future dates.

In tidal predictions, the accuracy of predicted values $\hat{y}(t)$ depends not only on the number of tidal components used in the computations, but also on the accuracy of their estimated amplitudes and phases. Assuming that we order the components according to their amplitudes, the more tidal components are included in an analysis, the higher the accuracy that can be achieved in tidal predictions. On the other hand, the accuracies of estimated amplitudes and phases of these tidal constituents, of which we would want to select as many as possible, are closely related to the length of the time series used in the estimation with the least squares method. If too many constituents are chosen for the analysis (in other words, if the time period over which observations are taken is too short), then either no solution would ensue, or an unstable solution would be obtained, in which the interference between and among tidal components with similar frequencies would be a detrimental factor.

This happens when two or more frequencies are too close together so that they cannot be resolved from the given length of the time series. As a rule of thumb, two tidal constituents of frequencies f_j and f_k can be separated, if their frequencies satisfy the relation [GODIN, 1972]:

$$n(f_j - f_k) \geq 1, \quad j \neq k. \quad (1)$$

This rule is called the Rayleigh criterion, where n represents the number of steps - typically hours - in a continuous sequence of observations.

From the above description, it is apparent that the number of tidal components that could be included in a tidal analysis really depends on the length of the available tide gauge record. Since tidal records of 'sufficient' length are not available at all tide gauge stations, it is usually impossible to obtain as many tidal constituents as we would wish to have to make a 'sufficient' good prediction. The problem then is that when the length of the collected tidal series is short, how many and which tidal constituents should be included in the analysis to give us the best predicted results. Before answering these questions, we should realize that since the tidal frequencies are distributed unequally in a few narrow bands, the selection of the tidal constituents to be included in both the analysis and prediction, is not a simple matter. Arbitrary selection of tidal constituents may cause either large departures of the prediction from reality, or cause the normal equations to become ill-conditioned (unstable), or even singular. Therefore, the selection of tidal constituents (from standard tidal tables) which fit different lengths of observational series is a critically important step, particularly for short tidal series.

As we can see from equation (1), by increasing the time period over which sea-level data are collected, more and more tidal constituents can be separated and thus included in the analysis. Then, the accuracy of tidal prediction will gradually increase. But, with adding new data and new tidal constituents, the normal equations of the harmonic analysis model have to be inverted repeatedly to update all the estimates. Usually, the matrix to be repeatedly inverted is quite large, so updating the estimates takes a good deal of CPU time.

The purpose of this work is to seek a method, that, while adopting the most detailed tidal model possible, would update harmonic results with a minimum computational effort.

CONVENTIONAL HARMONIC ANALYSIS

Let us, for the moment, forget the shallow water constituents and the non tidal effects in the oceans and consider the tide to be composed of astronomical tidal constituents only:

$$\begin{aligned}
 y(t_i) &= Z_0 + \sum_{j=1}^m H_j \cos(\omega_j t + \phi_j + \phi_j) = \\
 &= Z_0 + \sum_{j=1}^m (H_j \cos \phi_j \cos(\omega_j t + \phi_j) - H_j \sin \phi_j \sin(\omega_j t + \phi_j)), \quad (2) \\
 &= Z_0 + \sum_{j=1}^m (C_j \cos(\omega_j t + \phi_j) + S_j \sin(\omega_j t + \phi_j))
 \end{aligned}$$

where

$$C_j = H_j \cos \phi_j, \quad S_j = -H_j \sin \phi_j,$$

$t \in \{t_1, t_2, \dots, t_n\}$, $\omega t + \phi = k_1 \tau + k_2 s + k_3 h + k_4 p + k_5 N + k_6 Ps$ [MUNK and CARTWRIGHT, 1966], and t is the Greenwich mean time; H_j, ϕ_j are the unknown amplitudes and phase lags and Z_0 is the constant term. The following relations hold:

$$\forall j = 1, m: H_j = \sqrt{(C_j^2 + S_j^2)}, \quad \phi_j = \arctan\left(-\frac{S_j}{C_j}\right). \quad (3)$$

For the theoretical tide, the arguments $\omega_j t + \phi_j$ for individual tidal constituents are known from Doodson's harmonic development [DOODSON, 1923]. For the actual tide we have to estimate the unknown parameters Z_0, H_j, ϕ_j ($j = 1, 2, \dots, m$) from the series of measured values $y(t_i)$ ($i = 1, 2, \dots, n$) by the least squares methods. Once the parameters have been estimated, the values $\hat{y}(t_i)$ can be obtained from the above model (2); this is the tidal "prediction".

Matrix notation can now be used to rewrite equation (2) as:

$$Y = AX, \quad (4)$$

where $Y = (y(t_1), y(t_2), \dots, y(t_n))^T$ is the "observation vector", $X = (Z_0, C_1, S_1, \dots, C_m, S_m)^T$ is the "unknown vector", and the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & \cos(\omega_1 t_1 + \varphi_1) & \dots & \sin(\omega_m t_1 + \varphi_m) \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \cos(\omega_1 t_n + \varphi_1) & \dots & \sin(\omega_m t_n + \varphi_m) \end{bmatrix} \quad (5)$$

is called the Vandermonde (or design) matrix. The least squares solution of the above system of overdetermined equations (for $n > 2m+1$) is given by the following normal equations:

$$\hat{\mathbf{X}} = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{Y} \quad (6)$$

It is known from the theory of the least squares method [VANÍČEK and KRAKIVSKY, 1986] that the accuracy of the solution vector $\hat{\mathbf{X}}$ is estimated by the covariance matrix:

$$\hat{\mathbf{C}}_{\hat{\mathbf{X}}} = \frac{(\mathbf{Y} - \mathbf{A}\hat{\mathbf{X}})^T (\mathbf{Y} - \mathbf{A}\hat{\mathbf{X}}) (\mathbf{A}^T \mathbf{A})^{-1}}{n - (2m+1)} \quad (7)$$

The diagonal elements of $\hat{\mathbf{C}}_{\hat{\mathbf{X}}}$ represent the variances $\sigma_{\hat{x}_i}^2$ ($i=1, 2, \dots, 2m+1$) of the estimated parameters and off-diagonal elements are the covariances $\sigma_{\hat{x}_i \hat{x}_j}$ ($i, j=1, 2, \dots, 2m+1$) between pairs of parameters.

This method is widely used now in tidal operations because it is simple, yet it gives good enough predictions for most purposes.

LUMPED TIDAL CONSTITUENTS

As we have already stated, the number of constituents used in a tidal model depends strongly on the length of the observational series. If the length of observed series is long enough, all the harmonic tidal constituents of Doodson's development as well as shallow water constituents can be separated and thus included in the model. In other words, if the series is this long, then it makes no sense to introduce the sequential approach. But if the duration of observed series is much shorter, sometimes only a few months, weeks or even days, we wish to know which principal tidal constituents should be selected in the model to obtain the best predicted results. In other words, the question arises as to which selection of tidal constituents gives an optimal representation over whole tidal frequency spectrum. Making such a selection in tidal analysis is very difficult. No such composition captures the tidal energy distribution over all tidal frequency bands rigorously.

For constructing the sequential tidal model, we create representative lumped tidal constituents from individual astronomical and shallow water constituents, by

the least squares method. This smaller number of lumped tidal constituents, can be considered to be the best representation of tidal frequency bands. To demonstrate how to form these lumped tidal constituents, let us consider two pure astronomical tidal constituents with adjacent frequencies ω_1 and ω_2 ; lumped together the result must have a representative frequency ω^* located somewhere between ω_1 and ω_2 and a combined amplitude A^* somehow related to the amplitudes A_1 and A_2 of the original constituents (Fig. 1).

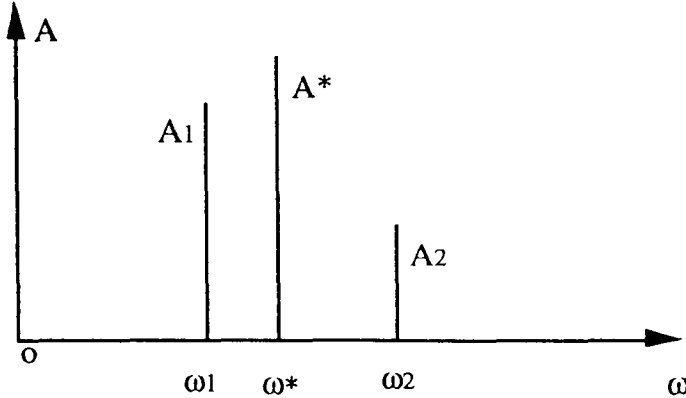


FIG. 1.- A representative frequency ω^* for ω_1 and ω_2 .

In our algorithm, the weighted average ω^* of the frequencies ω_1 , ω_2 of the two tidal constituents to be lumped together is used for the frequency of their lumped representative:

$$\omega^* = \omega_1 + \frac{\omega_2 - \omega_1}{A_1 + A_2} A_2, \quad (8)$$

where A_1 , A_2 are the respective theoretical amplitudes of the two constituents. The representative (lumped) amplitude A^* is defined as:

$$A^* = \sqrt{A_1^2 + A_2^2}. \quad (9)$$

These parameters ω^* and A^* are then used in the next lumping step as the representative values for the original constituents.

METHOD OF CONSTRUCTION OF LUMPED CONSTITUENT TABLES

The tidal potential contains about 400 constituents in Doodson's harmonic development [DOODSON, 1923], about 500 terms in CARTWRIGHT's spectral analysis results [CARTWRIGHT and TAYLOR, 1971] and about 1140 terms in QIWEN's logical deduction method for precision tidal analysis [QIWEN and TIANHANG, 1987].

In the construction of our lumped tidal tables, we have selected only 60 harmonic constituents whose theoretical relative amplitudes are larger than 500×10^{-5} (neglecting the fact that actual amplitudes may be significantly different, altered by tidal resonance and, of course, by latitude). Since the tidal energy is proportional to the squares of the amplitudes, the 60 principal constituents represent some 99.97% percent of total tidal energy. Beside the 60 principal tidal constituents, an additional 15 shallow water constituents have been also considered in our construction. Amplitudes of shallow water constituents change dramatically from place to place. In our computations, however, for a lack of any better information, we have assumed all their amplitudes, to equal to 600×10^{-5} . As shown in equation (8) and equation (9), theoretical amplitudes are only used in computing the lumped frequencies and their amplitudes.

When building up the design matrix, we re-write the observation equation (2) in such a way that the theoretical phase φ_j and the unknown phase lag ϕ_j are added together. Then they become part of the unknown parameters C_j^* and S_j^* , as follows:

$$\begin{aligned} \forall i=1, n: \quad y(t_i) &= Z_0 + \sum_{j=1}^m H_j \cos(\omega_j t + \varphi_j + \phi_j) = \\ &= Z_0 + \sum_{j=1}^m [H_j \cos(\varphi_j + \phi_j) \cos \omega_j t - H_j \sin(\varphi_j + \phi_j) \sin \omega_j t] = \\ &= Z_0 + \sum_{j=1}^m (C_j^* \cos \omega_j t + S_j^* \sin \omega_j t), \end{aligned} \quad (10)$$

where

$$\forall j = 1, m: \quad C_j^* = H_j \cos(\varphi_j + \phi_j); \quad S_j^* = -H_j \sin(\varphi_j + \phi_j). \quad (11)$$

Thus, the elements of the design matrix are expressed as functions of angular velocities ω_j and time t , regardless of the time origin of the analyzed series. It should be kept in mind, however, that the vector of unknown parameters C^* and S^* will change according to the choice of origin of time (usually Julian) used in forming the design matrix.

When the matrix of normal equations $N = A^T A$ is created, its elements have one of the following forms:

$$\sum_{i=1}^n \cos \omega_j t_i \cos \omega_k t_i, \quad \sum_{i=1}^n \cos \omega_j t_i \sin \omega_k t_i, \quad \sum_{i=1}^n \sin \omega_j t_i \sin \omega_k t_i \quad (12)$$

with the exception of the first row and first column, where the elements are as follows:

$$\sum_{i=1}^n \cos \omega_j t_i, \quad \sum_{i=1}^n \sin \omega_j t_i. \quad (13)$$

These elements can be computed much faster from the following equivalent expressions [BRONSHEIN and SEMENDYAYEV, 1979]:

$$\sum_{i=1}^n \cos \omega_j t_i \cos \omega_k t_i = \quad (14)$$

$$= \left\{ \begin{array}{l} \frac{\frac{n}{2} + \frac{\sin n \omega_j \cos [(n+1) \omega_j]}{2 \sin \omega_j}, \quad j=k \\ \frac{\sin [0.5n (\omega_j + \omega_k)] \cos [0.5 (n+1) (\omega_j + \omega_k)]}{2 \sin \frac{\omega_j + \omega_k}{2}} \\ + \frac{\sin [0.5n (\omega_k - \omega_j)] \cos [0.5 (n+1) (\omega_k - \omega_j)]}{2 \sin \frac{\omega_k - \omega_j}{2}}, \quad j \neq k, \end{array} \right.$$

$$\sum_{i=1}^n \cos \omega_j t_i \sin \omega_k t_i = \quad (15)$$

$$= \left\{ \begin{array}{l} \frac{\sin n \omega_j \sin [(n+1) \omega_j]}{2 \sin \omega_j}, \quad j=k \\ \frac{\sin [0.5n (\omega_j + \omega_k)] \sin [0.5 (n+1) (\omega_j + \omega_k)]}{2 \sin \frac{\omega_j + \omega_k}{2}} \\ + \frac{\sin [0.5n (\omega_k - \omega_j)] \sin [0.5 (n+1) (\omega_k - \omega_j)]}{2 \sin \frac{\omega_k - \omega_j}{2}}, \quad j \neq k, \end{array} \right.$$

$$\sum_{i=1}^n \sin \omega_j t_i \sin \omega_k t_i = \quad (16)$$

$$= \left\{ \begin{array}{l} \frac{n}{2} \frac{\sin n \omega_j \sin [(n+1) \omega_j]}{2 \sin \omega_j}, \quad j=k \\ \frac{\sin [0.5n (\omega_k - \omega_j)] \cos [0.5 (n+1) (\omega_k - \omega_j)]}{2 \sin \frac{\omega_k - \omega_j}{2}} \\ \frac{\sin [0.5n (\omega_k + \omega_j)] \sin [0.5 (n+1) (\omega_k - \omega_j)]}{2 \sin \frac{\omega_k + \omega_j}{2}}, \quad j \neq k, \end{array} \right.$$

$$\sum_{i=1}^n \cos \omega_j t_i = \frac{\sin [0.5n (\omega_j)] \cos [0.5 (n+1) \omega_j]}{\sin \frac{\omega_j}{2}} \quad (17)$$

$$\sum_{i=1}^n \sin \omega_j t_i = \frac{\sin [0.5n (\omega_j)] \sin [0.5 (n+1) \omega_j]}{\sin \frac{\omega_j}{2}} \quad (18)$$

Clearly, using the above expressions, the CPU time needed for constructing the matrix of normal equations is independent of the observation series length n . This is very useful in sequential tidal analysis, especially with a very long tidal series.

CORRELATION CRITERION FOR SEPARABILITY OF CONSTITUENTS

From the matrix of normal equations, we get some indication as to which two pairs of columns are likely to interfere with each other, i.e., which pair of tidal constituents is likely to be highly correlated. It is impossible, however, to determine the definite correlation values between any two adjacent constituents. For this purpose, it is necessary to invert the matrix and get the covariance matrix of the estimated coefficients (equation 7), which we will rewrite here as:

$$\hat{C} = \hat{\sigma}_o^2 (A^T A)^{-1} = \hat{\sigma}_o^2 \begin{bmatrix} Q_{11} & \cdot & \cdot & Q_{12m+1} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ Q_{2m+11} & \cdot & \cdot & Q_{2m+12m+1} \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \cdot & \cdot & \sigma_{12m+1} \\ \cdot & \sigma_2^2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \sigma_{2m+11} & \cdot & \cdot & \sigma_{2m+1}^2 \end{bmatrix} \quad (19)$$

where $\hat{\sigma}_o^2 = (Y-AX)^T (Y-AX)/(n-2m-1)$.

The correlation coefficient ρ_{ij} of any two estimated coefficients X_i, X_j (note that here X 's stand for the unknowns C 's and S 's) is given by [VANÍČEK and WELLS, 1972]:

$$\rho_{ij} = \frac{p_{ij}}{\sigma_i \sigma_j} \quad (20)$$

Clearly, we can also write:

$$\rho_{ij} = \frac{Q_{ij}}{\sqrt{Q_{ii}} \sqrt{Q_{jj}}} = \rho_{ji} \quad (21)$$

and to evaluate the correlation coefficient it is not necessary to know $\hat{\sigma}_o^2$. The following correlation matrix:

$$\mathbf{R} = \begin{bmatrix} 1 & \rho_{12} & \cdot & \rho_{1\ 2m+1} \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \rho_{2m+1\ 1} & \cdot & \cdot & 1 \end{bmatrix} \quad (22)$$

can be then calculated prior to having any observational series. This matrix \mathbf{R} depends only on the assumed length n of the time series, and the selected number m of constituents. Looking at the off-diagonal elements in the matrix \mathbf{R} , we can make a decision about which two (or more) tidal components are strongly correlated; these two (or more) can then be held responsible for potential ill-conditioning of the normal equations.

To determine which two adjacent tidal constituents are significantly correlated, we first take the 4 by 4 symmetrical submatrix of \mathbf{R} connected with these two constituents. This submatrix will look as follows:

$$\begin{bmatrix} 1 & \rho_{c_i s_i} & \rho_{c_i c_{i+1}} & \rho_{c_i s_{i+1}} \\ \rho_{s_i c_i} & 1 & \rho_{s_i c_{i+1}} & \rho_{s_i s_{i+1}} \\ \rho_{c_{i+1} c_i} & \rho_{c_{i+1} s_i} & 1 & \rho_{c_{i+1} s_{i+1}} \\ \rho_{s_{i+1} c_i} & \rho_{s_{i+1} c_{i+1}} & \rho_{s_{i+1} c_{i+1}} & 1 \end{bmatrix} \quad (23)$$

From numerical experiments we have established that

$$\rho_{c_i c_{i+1}} \approx \rho_{s_{i+1} c_{i+1}} \quad \rho_{c_{i+1} s_{i+1}} \approx \rho_{s_i c_{i+1}} \quad (24)$$

From further numerical experiments we have determined that the quantity

$$\rho_{i,i+1} = \frac{1}{2} \sqrt{ (|\rho_{c_i c_{i+1}}| + |\rho_{s_{i+1} c_{i+1}}|)^2 + (|\rho_{c_{i+1} s_{i+1}}| + |\rho_{s_i c_{i+1}}|)^2 } \in (0,1) \quad (25)$$

decreases monotonically with increasing n , and is thus suitable to use as a correlation criterion. Somewhat arbitrarily, we have selected a value of 0.985 to serve as a limiting value for deciding if two adjacent constituents are correlated, i.e., if they are separable or not (for $\rho_{i,i+1} \leq 0.985$ all the correlation coefficients in matrix (23) are smaller in absolute value than 0.95). For any two adjacent constituents, the value n for which they become separable can then be determined.

Representative lumped constituents, obtained as described above, are given in the individual boxes in Tables 1 to 3. These tables show how the lumping works for diminishing length of observational series, when the series is shortened in successive steps. Looked at from the other perspective, i.e., considering the series as growing in length, these tables show the separability of constituents. Based on these tables, detailed time schedule for the tidal constituent separability has been designed and included in our computer program for sequential tidal analysis.

THE SEQUENTIAL MATHEMATICAL MODEL

In applying the sequential technique to tidal analysis, we have to have the first estimates of unknown parameters which are obtained from an initial tidal harmonic analysis. In the next stage these initial estimates are updated by computing corrections to the earlier results as functions of previous estimates. The information made available for subsequent use consists of the estimated amplitudes, phase lags and their covariance matrix.

Table 1 - Separability of Long Period Tidal Constituents

No.	Darw. Symb.	Hours Fn	(1250d) 30000	(500d) 12000	(167d) 4000	(42d) 1000	(25d) 600	(12.5d) 300	(5d) 120
1	S_o	0.0022064		0.0080 (800)					
2	S_a	0.0410667			0.04676 (9863)				
3	S_{sa}	0.0821373				0.27080 (12979)			
4		0.4715211					0.53272 (8437)		
5		0.5421683	0.54424 (8271)						
6	M_m	0.5443747		0.54438 (8288)					
7		0.5465811						0.73682 (21402)	
8		1.0158958							
9		1.0887489						1.09226 (170127)	
10	M_t	1.0980330	1.09868 (16948)		1.09842 (16962)				
11		1.1002398		1.09881 (16951)					
12		1.1024456							
13		1.5695543							
14		1.6424047		1.64305 (3243)				1.6321 (3292)	
15		1.6446142							0.85617 (21654)

Table 2 -- Separability of Diurnal Tidal Constituents

No.	Darw. symb.	Hours	(1250d)	(500d)	(167d)	(42d)	(25d)	(12.5)	(4d)	(2d)
		Fh	30000	12000	4000	1000	600	300	96	48
1		12.854286								
2		12.927140								
3		13.396455								
4	Q₁	13.398661		13.39834 (7345)						
5		13.471515								
6		13.940829								
7	O₁	13.943035		13.94268 (88361)						
8	MP	14.025173								
9		14.487411		14.48949 (-1254)						
10		14.492052								
11	M₁	14.496694		14.49706 (-3023)						
12		14.498900								
13		14.569548								
14		14.917865								
15	P₁	14.958931								
16		15.038862								
17	K₁	15.041069		15.04129 (53509)						
18		15.043275								
19		15.123206								
20		15.512590								
21	J₁	15.585444		15.58581 (-3023)						
22		15.587649								
23	SO₁	16.056961								
24		16.139099		16.13996 (-1927)						
25	O₀₁	16.141312								

Table 3 -- Separability of Semidiurnal Tidal Constituents

No.	Darw. Symb.	Hours Fn	(500d) 12000	(167d) 4000	(42d) 1000	(25d) 600	(12.5) 300	(5d) 120	(4d) 96
1	NK _{2S}	26.961594							
2	MNS	27.423828	27.42383 (901)						
3	OP ₂	27.423834							
4	2N ₂	27.895355							
5		27.968216							
6		28.435089							
7		28.437531							
8	N ₂	28.439728							
9	V ₂	28.512589							
10		28.901962							
11		28.981903							
12	M ₂	28.984100							
13		29.455627							
14	L ₂	29.528473							
15		29.533127							
16		29.537766							
17	T ₂	29.958939							
18	S ₂	30.000000							
19	K ₂	30.082138							
20		30.626511							


```

    graph TD
      1[1 NK2S 26.961594] --> 2[2 MNS 27.423828]
      1 --> 3[3 OP2 27.423834]
      2 --> 4[4 2N2 27.895355]
      2 --> 5[5 27.968216]
      3 --> 6[6 28.435089]
      3 --> 7[7 28.437531]
      4 --> 8[8 N2 28.439728]
      4 --> 9[9 V2 28.512589]
      5 --> 10[10 28.901962]
      6 --> 11[11 28.981903]
      6 --> 12[12 M2 28.984100]
      7 --> 13[13 29.455627]
      8 --> 14[14 L2 29.528473]
      8 --> 15[15 29.533127]
      8 --> 16[16 29.537766]
      9 --> 17[17 T2 29.958939]
      9 --> 18[18 S2 30.000000]
      9 --> 19[19 K2 30.082138]
      10 --> 20[20 30.626511]
      11 --> 20
      12 --> 20
      13 --> 20
      14 --> 20
      15 --> 20
      16 --> 20
      17 --> 20
      18 --> 20
      19 --> 20
      20 --> 20
  
```

In our sequential tidal model algorithm we distinguish between two different update modes:

- (i) adding only new observations;
- (ii) adding both new observations and new tidal constituents.

Throughout the development of the algorithm (as well as the program based on the algorithm), we restrict ourselves to the rigorous approach, but we will include some discussion concerning approximate approaches for certain situations.

The original mathematical model for tidal harmonic analysis is (cf. eqn. 4):

$$\mathbf{A}_1 \mathbf{X}_1 = \mathbf{L}_1 \quad (26)$$

where $\mathbf{L}_1 = (L_1, L_2, \dots, L_n)^T$ is the data vector, \mathbf{A}_1 is the design matrix, and $\mathbf{X}_1 = (X_1, X_2, \dots, X_{2m+1})^T$ is the unknown parameter vector. The least squares solution of the above system of over-determined linear equations is given by normal equations as [VANÍČEK and KRAKIVSKY, 1986]:

$$\hat{\mathbf{x}}_1 = \mathbf{N}_1^{-1} \mathbf{A}_1^T \mathbf{L}_1 = (\mathbf{A}_1^T \mathbf{A}_1)^{-1} \mathbf{A}_1^T \mathbf{L}_1 \quad (27)$$

The sequential updating starts with the acquisition of additional data. Let us assume that a batch of data, consisting of 1 to n_1 new values, becomes available. We note that the size n_1 of the batch (it may be as small as 1!) should be selected beforehand according to what use the results of the analysis are going to be put to.

When adding the new batch of data to the existing series, two or more tidal constituents may become separable. If this is the case, then the separable constituent present in the previous analysis - it could have been a lumped constituent, of course - is replaced by its separate component. This case is referred to as ii) above. We shall first discuss the more simple scenario i), when no new tidal constituents appear in the sequential step.

ADDITION OF NEW OBSERVATIONS

If only new observations $\mathbf{L}_2 = (L_{n+1}, L_{n+2}, \dots, L_{n+n_1})^T$ are added, the observation equations (26) become:

$$\begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{bmatrix} (\mathbf{X}_1^{(1)} + \delta \mathbf{X}_1^{(2)}) = \begin{bmatrix} \mathbf{L}_1 \\ \mathbf{L}_2 \end{bmatrix} \quad (28)$$

and the new solution is given as:

$$\hat{X}_1^{(1)} + \delta \hat{X}_1^{(2)} = N_2^{-1} (A_1^T L_1 + A_2^T L_2) \quad (29)$$

Here, the matrix N_2 of normal equations has the same dimensions as N_1 in equation (27) and it read as follows:

$$N_2 = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}^T \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = A_1^T A_1 + A_2^T A_2 = N_1 + \Delta N_2 \quad (30)$$

The inversion of this matrix can be obtained from the following rigorous sequential expression [MORRISON, 1976]:

$$N_2^{-1} = (N_1 + \Delta N_2)^{-1} = N_1^{-1} - N_1^{-1} A_2^T [I + A_2 N_1^{-1} A_2^T]^{-1} A_2 N_1^{-1} \quad (31)$$

Here, I is the identity matrix, and $\Delta N_2 = A_2^T A_2$ can be considered the perturbation of the set of the original normal equations due to the added observations. From expression (31), we see that the matrix $I + A_2 N_1^{-1} A_2^T$ to be inverted has a dimensions of n_1 by n_1 , where n_1 is the number of added observations.

If n_1 is large, we need to invert a large matrix which would eliminate one of the important advantages of the sequential approach. In practice, the number of added observations n_1 should be small. If we let, for example, $n_1=1$, meaning that only one new observation is added at a time, the matrix above degenerates into a scalar:

$$(I \ A_2 \ N_1^{-1} \ A_2^T) = Q. \quad (32)$$

equation (31) is then written as:

$$N_2^{-1} = (N_1 + \Delta N_2)^{-1} = N_1^{-1} - \frac{1}{Q} N_1^{-1} A_2^T A_2 N_1^{-1} = N_1^{-1} - \frac{1}{Q} N_1^{-1} \Delta N_2 N_1^{-1} \quad (33)$$

Obviously, when adding a single new observation at a time, no additional matrix needs be inverted.

Before the rigorous complete sequential solution is given, let us introduce an important approximate formula for matrix inversions which may be useful in some cases where the original matrix is huge, and the number n_1 of observations added is so large that the rigorous inversion would be too time consuming. The approximate expression reads [CRAYMER and VANÍČEK, 1989].

$$N_2 = (N_1 + \Delta N_2)^{-1} = N_1^{-1} - N_1^{-1} \Delta N_2 N_1^{-1} \quad (34)$$

This approximation may be used when

$$\|\Delta N_2\| < \|N_1\| \quad (35)$$

where $\|\cdot\|$ denotes a norm [VANÍČEK and KRAKIWSKY, 1986]. There are several ways to compute a matrix norm, and we adopt here the most commonly used quadratic norm. The formulation is given as:

$$\|T\| = \left\{ \sum_{ij} |T_{ij}|^2 \right\}^{\frac{1}{2}} \quad (36)$$

where T_{ij} are the elements of matrix T .

To conclude: if no new tidal constituents are added, the new (sequential) solution is given by:

$$X_1^{(2)} = X_1^{(1)} + \delta X_1^{(2)} \quad (37)$$

$$\delta X_1^{(2)} = N_1^{-1} A_2^T L_2 - N_1^{-1} F N_1^{-1} (A_1^T L_1 + A_2^T L_2) \quad (38)$$

where the matrix F is given as

$$F = \begin{cases} -A_2^T (I + A_2 N_1^{-1} A_2^T)^{-1} A_2 & \text{(general case)} \\ -\frac{1}{Q} \Delta N_2 & \text{for } n_1 = 1 \\ -\Delta N_2 & \text{when } \|\Delta N_2\| < \|N_1\|. \end{cases} \quad (39)$$

SEPARATION OF TIDAL CONSTITUENTS

The separation of tidal constituents is done in 3 steps.

1) In the first step we consider the complete observation vector $L = (L_1, \dots, L_{n+n_1})$ and solve for the same constituents as in the previous step. This step is thus identical to the one described in the previous paragraph. We write the result of this step as

$$\mathbf{X}^{(2)} = \mathbf{X}^{(1)} + \delta \mathbf{X}^{(1)} \quad (40)$$

2) In the second step, we leave the number of observations as it is, i.e. at $n+n_1$, and solve for $2(m-i)+1 = 2m - 2i + 1$ parameters, to erase the i constituents (which possess $2i$ parameters) to be separated; typically, i will be equal to 1. We thus first split the vector of parameters $\mathbf{X}^{(2)}$ as follows:

$$\mathbf{X}^{(2)} = \begin{bmatrix} \mathbf{X}_1^{(2)} \\ \mathbf{X}_2^{(2)} \end{bmatrix} \quad (41)$$

and proceed to forget about the $2i$ -dimensional vector \mathbf{X}_2 . Then, the first part of the vector of parameters, $\mathbf{X}_1^{(2)}$, must be corrected for the effect of discarding the $2i$ parameters. This is done by computing a correction

$$\delta \mathbf{X}_1^{(2)} = - \mathbf{M}_{12} [(\mathbf{M}_{22})^{-1} \mathbf{M}_{21} \mathbf{A}_1^T - \mathbf{A}_2^T] \mathbf{L} \quad (42)$$

where \mathbf{M}_{ij} are submatrices of the complete covariance matrix $(\mathbf{N}^{(2)})^{-1}$ of \mathbf{X} . This matrix can be written as:

$$(\mathbf{N}^{(2)})^{-1} = \begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1^T \mathbf{A}_1 & \mathbf{A}_2^T \mathbf{A}_1 \\ \mathbf{A}_1^T \mathbf{A}_2 & \mathbf{A}_2^T \mathbf{A}_2 \end{bmatrix}^{-1} \quad (43)$$

Here $\mathbf{A}_1, \mathbf{A}_2$ are the two parts of the original design matrix corresponding to the two parts of \mathbf{X} . Finally we evaluate $\mathbf{X}_1^{(3)}$ from

$$\mathbf{X}_1^{(3)} = \mathbf{X}_1^{(2)} + \delta \mathbf{X}_1^{(2)} \quad (44)$$

3) In the third step, we again leave the number of observation unchanged at $n + n_1$, and solve for $2(m+2i) + 1$ parameters, having added the $4i$ new parameters. To do this, the vector of existing parameters is first augmented by $4i$ -dimensional vector

$$\mathbf{X}_3^{(3)} = \mathbf{0} \quad (45)$$

and we write

$$\mathbf{X}^{(3)} = \begin{bmatrix} \mathbf{X}_1^{(3)} \\ \mathbf{X}_3^{(3)} \end{bmatrix} \quad (46)$$

Now, once again, the new vector of parameters has to be corrected for the effect of the added parameters. The correction is computed from the following expression:

$$\begin{aligned} \delta \mathbf{X}^{(3)} &= \begin{bmatrix} \delta \mathbf{X}_1^{(3)} \\ \delta \mathbf{X}_3^{(3)} \end{bmatrix} = (\mathbf{N}^{(2)})^{-1} \begin{bmatrix} \mathbf{A}_1^T \\ \mathbf{A}_3^T \end{bmatrix} \mathbf{L} \\ &= \begin{bmatrix} (\mathbf{N}_{11})^{-1} \mathbf{N}_{12} \mathbf{D}^{-1} \mathbf{N}_{21} (\mathbf{N}_{11})^{-1} & -(\mathbf{N}_{11})^{-1} \mathbf{N}_{12} \mathbf{D}^{-1} \\ -\mathbf{D}^{-1} \mathbf{N}_{21} (\mathbf{N}_{11})^{-1} & \mathbf{D}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{A}_1^T \\ \mathbf{A}_3^T \end{bmatrix} \mathbf{L} \end{aligned} \quad (47)$$

where \mathbf{N}_{11} stands for the $\mathbf{N}^{(2)}$ from the previous step.

$$\mathbf{D} = (\mathbf{N}_{22} - \mathbf{N}_{21} (\mathbf{N}_{11})^{-1} \mathbf{N}_{12}) \quad (48)$$

and $\mathbf{N}_{12}, \mathbf{N}_{21} = \mathbf{N}_{12}^T, \mathbf{N}_{22}$ are the submatrices of the current complete matrix of normal equations:

$$\mathbf{N}^{(3)} = \begin{bmatrix} \mathbf{N}_{11} & \mathbf{N}_{12} \\ \mathbf{N}_{21} & \mathbf{N}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1^T \mathbf{A}_1 & \mathbf{A}_3^T \mathbf{A}_1 \\ \mathbf{A}_1^T \mathbf{A}_3 & \mathbf{A}_3^T \mathbf{A}_3 \end{bmatrix} \quad (49)$$

The \mathbf{A} matrices are, of course, the design matrices. Then, we get finally

$$\mathbf{X}^{(4)} = \mathbf{X}^{(3)} + \delta \mathbf{X}^{(3)} \quad (50)$$

This is the required new vector of dimensional $2(m+2i)+1$ of the augmented parameters best fitting to the updated $n+n_1$ - dimensional vector of observations.

Writing the result of the above 3 steps together, we obtain

$$\mathbf{X}^{(4)} = \begin{bmatrix} \mathbf{X}_1^{(1)} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \delta \mathbf{X}_1^{(1)} + \delta \mathbf{X}_1^{(2)} + \delta \mathbf{X}_1^{(3)} \\ \delta \mathbf{X}_3^{(3)} \end{bmatrix} \quad (51)$$

This vector then becomes again $\mathbf{X}^{(1)}$ in the next sequential step, be it either the simple addition of observations, or addition of observations combined with the next separation of tidal constituents. It is interesting to see that the additional matrix inversions for sequential separation of constituents are only at most of the size of the number of added parameters.

In the sequential process, the inverse of the entire matrix of normal equations, i.e., the covariance matrix of estimated parameters, in the current step must be available in the next step, to obtain the rigorous sequential solution.

TESTING OF ACCURACY OF FIT

To test the performance of our sequential algorithm, we had generated a synthetical hourly series consisting of the most dominant 60 theoretical (astronomical) constituents and 15 shallow water constituents. We then analyzed this series, starting with the first 100 values and proceeding till 300 hours (i.e., 12.5 days) were reached, using a step of two hours. From the lumped constituent tables, the program selected 12 lumped constituents to be fitted to the first 100-value series and ended up fitting 21 lumped constituents to the whole 300-value series. At each step we plotted the relative RMS error defined as $\sigma_i / \bar{\zeta}$, where

$$\forall i = 1, 300: \quad \sigma_i = \left[\sum_{j=1}^{n_i} (\zeta_j - \hat{\zeta}_j)^2 / (n-2m-1) \right]^{\frac{1}{2}} \quad (52)$$

$$\bar{\zeta} = \left[\sum_{j=1}^n \zeta_j^2 / n \right]^{\frac{1}{2}}. \quad (53)$$

with ζ_j being the generated values and $\hat{\zeta}_j$ being the estimated values. This relative RMS is shown in Figure 2, where the symbols (13), (14), ... (21) indicate the number of lumped tidal constituents used at any particular time.

The shape of the curve demonstrates that when new tidal constituent are added (really, when used lumped constituents are separated), the relative accuracy of the fit increases. During the time interval when the number of tidal constituents in the model is fixed, the relative accuracy decreases until the next separation of constituents occurs. It implies that before adding the next constituent at a certain

stage, the tidal model finds it more and more difficult to fit properly the current data series. Intuitively, this behaviour makes a good sense.

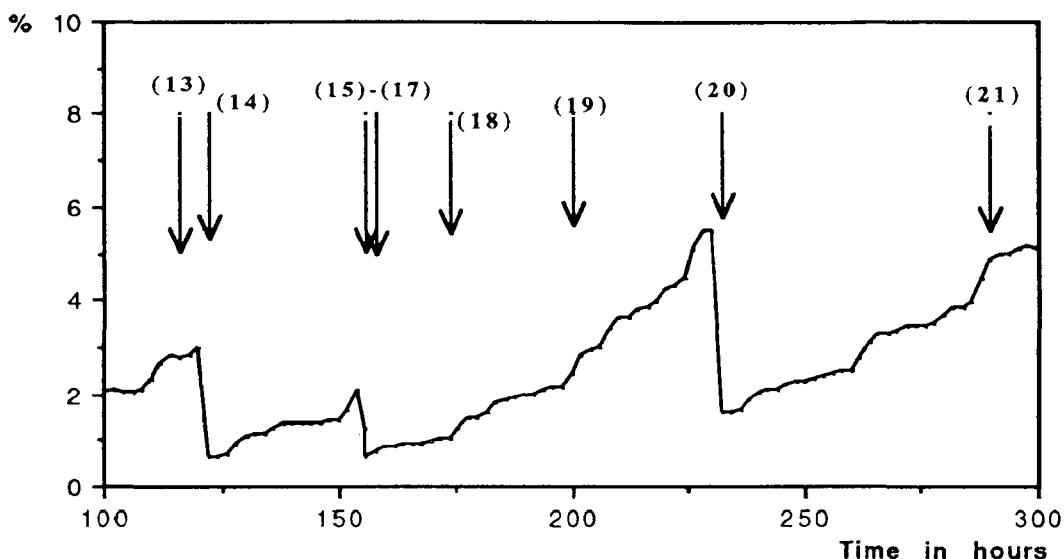


FIG. 2.- Relative RMS of the Sequential Analysis.

For comparison, we also give the RMS (σ) curve of the real tide-gauge data analysis at Halifax (Fig. 3). It shows that the values of standard deviations of the fit also generally decrease when a new constituent is added to the model. The situation in this case is more complicated however, because of the presence of non-tidal signals in the data. Thus, Figures 2 and 3 cannot be compared directly. Some modeling of non-tidal contributions would be needed to improve the performance of the sequential algorithm with real data.

COMPUTATION SPEED TESTING

The tidal harmonic analysis results at permanent tide gauges should be kept up-to-date to maintain the quality of tidal prediction at any given time. To do this, large systems of linear equations have to be solved repeatedly. This requires a lot of CPU time, which, in turn, increases the cost. It is thus of natural interest, to determine just how much faster the analysis can be performed using the sequential approach. The comparison of the time consumption of the traditional harmonic analysis with that of the sequential approach is given in Figure 4.

The difference in the CPU time consumption (including solving the initial system of normal equation in the sequential method) for obtaining solutions with increasing number of unknown parameters (on the UNB mainframe IBM computer) is seen very clearly. For instance, if the tidal model contains 7, constituents (15

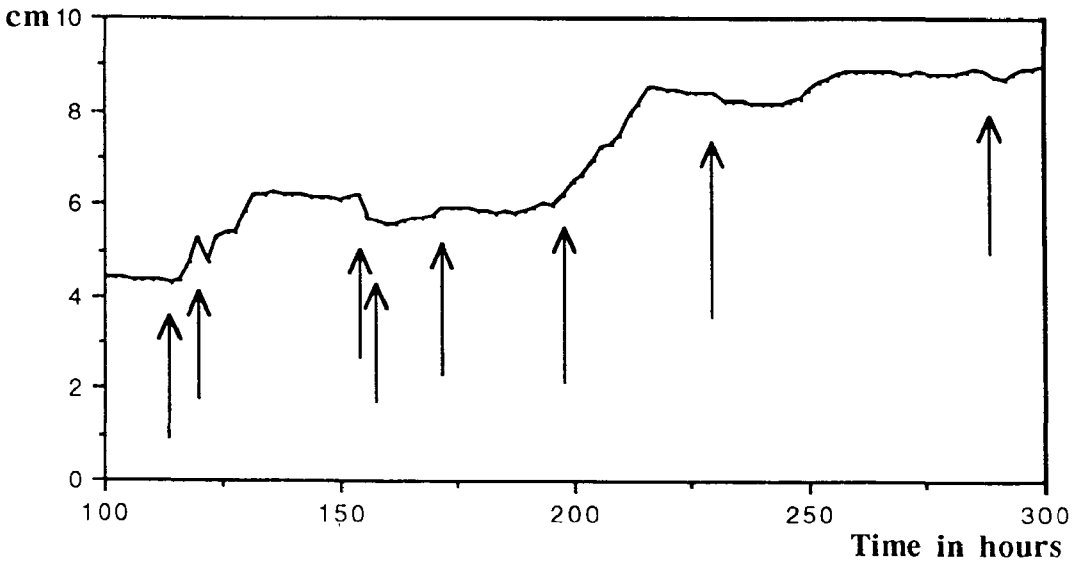


FIG. 3.- The RMS of the Sequential Analysis at Halifax.

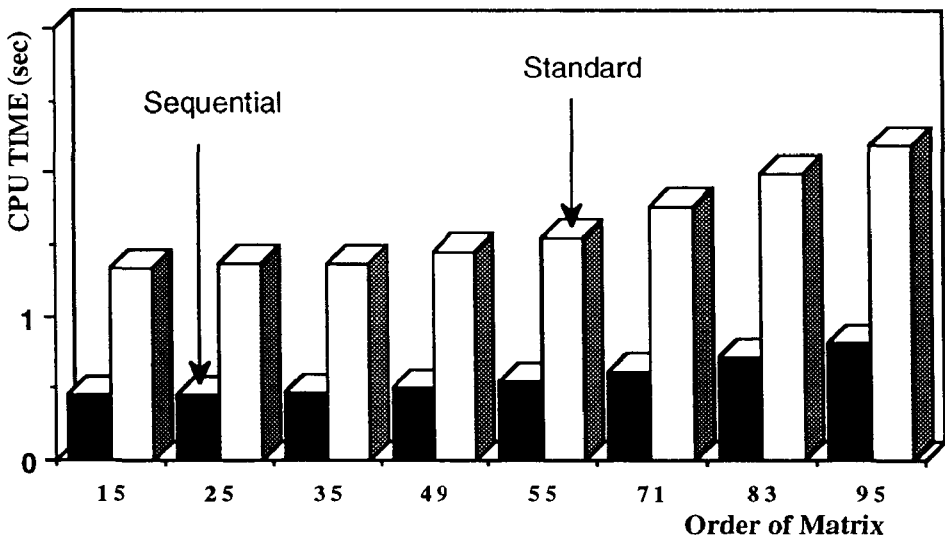


FIG. 4.- CPU Time Consumption of the Two Methods.

unknown parameters), solving the pertinent 15 normal equations to get updated solutions, the standard method spends 1.34 seconds of CPU time, while the sequential method spends 0.45 seconds - a saving of 0.89 seconds, i.e., 66%. If the size of the system of equations is increased to 95, the difference of CPU time needed by the two methods increases to 1.38 seconds, i.e., 63%. With further increases in the number of needed constituents, the CPU time saving increases progressively.

COMPARISON BETWEEN USING PURE AND LUMPED CONSTITUENTS

The lumped tidal constituent tables discussed above are based on the astronomical tidal constituents and created by the least squares method, in which the covariance matrix of the estimated constituents' amplitudes and phases is inspected by using a specific correlation criterion. With the lumped constituent tables, we establish a standard model that includes as many tidal components as possible with the limited length of observational series, while assuring that well-conditioned normal equation matrix results in any of the steps of the sequential algorithm. As we have mentioned earlier, the lumped tidal constituents can be considered as a good representation of the pure astronomical and shallow water constituents when the time series is short. For demonstrating the differences between using the two kinds of constituents, pure and lumped, two data series, a synthetic one and the observed data series at Halifax, were analysed. The resulting σ_i of the respective fits computed by equation (48) are shown in Table 4

Table 4 - Standard deviation of analyses with pure and lumped constituents

	n (hours)	Observed series (cm)	Synthetic series (cm)
Pure (Astronomical)	40	±5.826	±1.414
Tidal	70	±5.213	±1.303
Constituents	100	±5.046	±0.264
Lumped	40	±4.599	±0.408
Tidal	70	±3.743	±0.151
Constituents	100	±4.410	±0.365

It can be seen that when the series is not very long, the analysis with lumped constituents yields generally more accurate results than the one with the pure constituents. The only exception is found for the longest analyzed stretch of the synthetical data series ($n = 100$ hours). The reason is that the lumped constituents used in the analysis contain some shallow water contributions. When the length of the series is increased, the effect of these contributions on the lumped constituents becomes more and more apparent and the misfit of the fitted series to the synthetical series (generated from purely astronomical constituents) becomes more and more obvious. It may be assumed that if the lumped constituent tables were constructed without considering the shallow water effects, the accuracy of the analysis (using these lumped constituents) would also be higher.

As one may expect, prediction with lumped tidal constituents gives also a higher accuracy than that with pure astronomical constituents. This can be seen from numerical results listed in Table 5, constructed for $n = 100$ hours and predicted for another 30 hours. The standard deviations were computed from equation (48), where estimated and predicted values were computed with the values of the analyzed/predicted series.

Table 5 - Estimations and predictions with pure and lumped constituents (in centimeters)

	Lumped constituents	Pure constituents
Stand. deviations for estimates	4.410	5.046
Stand. deviations for predictions	12.881	15.291

When the length of the time series is increased, the results by using the two kinds of constituents get closer together, until the difference between them completely disappears. This happens, when the series becomes sufficiently long so that all the lumped constituents can be separated into their pure constituents.

CONCLUSIONS

The sequential tidal harmonic analysis proposed in this study can be used to provide up-to-date information for ocean tidal predictions in real time. Once new hourly observations become available, updated results (estimated new amplitudes and phase lags, and their standard deviations) can be obtained with very little CPU time expenditure, as the solution time is only weakly dependent on how many tidal constituents are included in the tidal model. If desired, the prediction values ζ can be naturally computed in each sequential step.

For obtaining accurate enough results for shorter series lumped tidal constituent tables have been constructed. This has been done by using the correlation matrix for estimated tidal amplitudes and phases and by applying a specific criterion for maximum allowable correlation. If the need arises, these tables can be recomputed for a different criterion. It appears that lumped tidal constituents are a realistic representation of pure astronomical tidal constituents over all tidal frequency bands with observational series of a limited length.

The program for sequential tidal harmonic analysis is recommended for use with tide gauges equipped with micro-computers. If the initial tidal analysis contains a large number of constituents, the covariance matrices and unknown parameters can be calculated beforehand in a larger computer. This solution (unknown parameters can be calculated beforehand in a larger computer. This solution (unknown parameters and their covariance matrix) may be input as a file on a diskette for sequential computations in the micro-computer. At the tide gauge, only sequential computations for the updating of the unknown and future predictions would be carried out.

To increase the accuracy of tidal analysis, the non-tidal effects, which result from long periodic causes such as glacial melts, crustal movement, river discharge and several important meteorological variations, should be filtered out from data before performing the sequential tidal analysis. Alternatively, these non-tidal effects could be modeled within the sequential analysis.

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