International Hydrographic Review, LXXV(2), September 1998

MAPPING THE FOOT OF THE CONTINENTAL SLOPE WITH SPLINE-SMOOTHED DATA USING THE SECOND DERIVATIVE IN THE GRADIENT DIRECTION

by John O. BENNETT [1]

Paper published as OCS Report, MMS 97-0018, 1998.

Abstract

The United States National Oceanic and Atmospheric Administration's (NOAA) ETOPO5 worldwide digital bathymetric dataset has been in the public domain for some years. Because it is noisy, it has not found much use in oceanography. A bi-cubic spline approach is used to smooth out the noise and represent the data as an explicit mathematical function, thus making it useful in many areas of oceanography. This method requires the data to have a rectangular grid.

This report[2] gives an effective approach for ETOPO5 data's bi-cubic spline representation and smoothing. It presents a new procedure designed to determine the Foot of the Continental Slope (FCS). This procedure is in accord with The United Nations Law of the Sea (LOS) article 76, section 4.b legal definition of the FCS, which is "the rate of maximum change of the gradient at its base". This explicit mathematical function can also be used to refine the grid. This function can also be differentiated exactly. One may compute from this function, at any point, the second derivative in the normalized gradient direction. The resulting surface is called for brevity "Surface of Directed Gradient" (SDG). The location of the crest of its highest ridge is a good approximation of the FCS. This approach gives an accurate mathematical representation of the LOS Convention's legal description of the FCS as stated above. The SDG technique is used to compute the FCS for the U.S. Atlantic coast.

The FCS computed by the SDG method is compared to the FCS computed by the surface of maximum curvature approach that is in general use.

[1] U.S. Department of the Interior, Minerals Management Service, Resource Evaluation Division, Herndon, Virginia.

[2] This report does not necessarily represent the methodology the United States will employ in defining the outer limit of its continental shelf.

Introduction

The Minerals Management Service (MMS) is the bureau in the Department of the Interior that is responsible for managing the mineral resources on the Federal Outer Continental Shelf (OCS) and Exclusive Economic zone (EEZ). The mineral resources include, but are not limited to, oil, gas, sulphur, sand, gravel, phosphorites, manganese, cobalt, and heavy minerals. The MMS, therefore, has an interest in defining the Foot of the Slope and outer limits of the EEZ. The United Nations Law of the Sea (LOS) gives a legal definition of the Foot of the Continental Slope (FCS). The FCS extends a nation's mineral rights past the 200-nautical mile Exclusive Economic zone (EEZ) where the FCS is beyond 200 nautical miles from that country's coast. This is true for some parts of the U.S. Atlantic coast. The accurate location of the FCS is important to any coastal country whose FCS extends beyond the 200-nautical mile EEZ. As new technology allows for deeper offshore drilling and mining, the location of the FCS will become more crucial. Location of the FCS can also be of importance in boundary disputes between coastal countries.

The United Nations Convention on the LOS states, in Article 76(4)(b), that the "foot of the continental slope (FCS) shall be determined as the point of maximum change in the gradient at its base". See Figure 1 for a cross-section showing the location of the FCS. To follow the legal description of FCS as cited above, one must proceed in the direction of the gradient at each point (x,y) of the digital bathymetric dataset. The computational procedure, as presented in this report, generates the surface by computing the second derivative in the normalized gradient direction of the smoothed function. This is the same as the Rayleigh Quotient (R.Q.) of the Hessian (**H**) of the smoothed function evaluated at the gradient. The resulting surface is called the "Surface of Second Derivative in the Gradient Direction" or for brevity the "Surface of Directed Gradient" (SDG). The location of the crest of the highest ridge of this surface is a good approximation to the location of the FCS. This procedure is then used to compute the FCS for an area covering most of the U.S. Atlantic coast using a spline-smoothed version of NOAA'S ETOPO5 bathymetric dataset. For a discussion of splines see de BOOR (1978) or SCHUMAKER (1981).

VANICEK et al. (1994) determine the FCS by computing the "Surface of Maximum Curvature" (SMC). Because the SMC approach requires a scaling of the x, y, and z axes to the same units to give satisfactory results. After the scaling and the FCS is obtained, the results need to be scaled again to return it to the proper aspect ratio. After scaling twice, the SMC obtains essentially the same results as those obtained by the SDG in locating the FCS. The SMC, as outlined by VANICEK, when implemented on the original ETOPO5 data with the axes being x: degrees longitude y: degrees latitude, and z: metres, did not give a satisfactory FCS.

The notation in labeling the grid intervals of the figures is an issue throughout this report. The original coordinate units for the ETOPO5 data are: x: degrees longitude, y: degrees latitude, and z: metres measured below sea level. These are measured on a sphere. In displaying the data and results of this report, a flat surface is used. One degree of latitude on the y-axis is approximately 60 nautical miles or 5 minutes of latitude are approximately 5 nautical miles. When converting to nautical miles from degrees on the x or longitudinal axis, the farther the distance is from the Equator, the smaller is a degree of longitude according to the rule:

$$1 \text{ degree longitude (at latitude } \theta) = (60 \text{ nautical miles}) \times \cos(\theta).$$

with θ being 90 degrees at the Equator and 0 degrees at the North Pole.

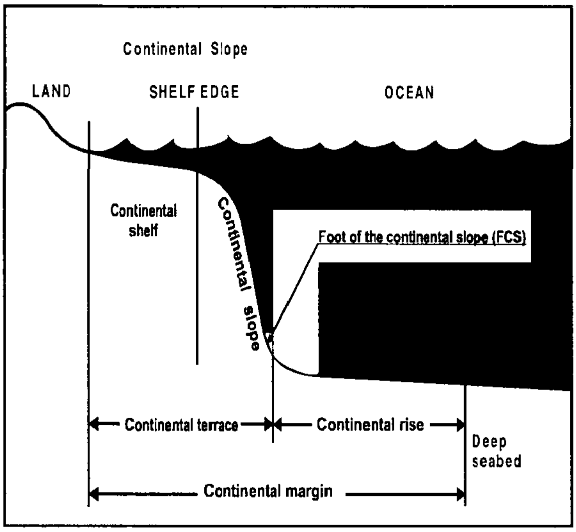

FIG. 1.- The foot of the Continental Slope (FCS) as located on cross-section of the continental shelf.

For example, when a grid interval of 5 nautical miles is referred to, it means 5 nautical miles on the *y*-axis and 5 x cos(36.5) = 4.02 nautical miles on the *x*-axis. (See Figs. 4-6) The 36.5° is an average latitude over the area. Similarly, there is a 2.5-nautical mile grid interval in the *y* direction and a 2.5 x cos(36.5°) = 2.001-nautical mile grid interval in the *x* direction. (See Fig; 11). Grid intervals in kilometres are given exactly for the *x* and *y* axes. (See Figs. 13 and 15).

SURFACE OF THE SECOND DERIVATIVE IN THE GRADIENT DIRECTION (SDG)

The theoretical mathematical derivation presented in this section was written by Carl de BOOR (1995, On determining the Foot of the Continental Slope, private communication (adapted)). The official United Nation's definition of the FCS seems to be made with a univariate image in mind. One previous approach to compute the FCS uses the location of the highest ridge of the SMC. This approach seems to interpret the "maximum change in the gradient" to mean "maximum curvature" with a surface. This interpretation of the "maximum in the rate of change of the gradient" to be the same as a maximum in the curvature of the seafloor ignores the legal definition, which suggests a profile or curve. The curvature will have a maximum at a well-defined foot; however, it can be large at other points also.

To comply with the legal definition, the approach should be one of a cross-section profile of the continental shelf in transition from the relatively flat continental shelf dropping to a steeply dipping slope to the rather flat rise of the continental rise or abyssal plain. See Figure 1. The narrow region where the continental slope meets the continental rise is the FCS. It is characterized by the rapid change of the gradient from a steep slope to one of being almost flat. In mathematical terms, curvature in that small region is maximally positive. Which profile should one choose to comply with the legal definition at any point (x,y)? The answer is the profile determined by the gradient at (x,y).

An approach representative of the legal definition of the FCS would be obtained as follows: Let $S(x,y)$ (See app. A, eq. A2) be a bi-cubic spline function generated by Matlab M-file TCSAPS and represent the smoothed bathymetric data:

$$Z(x_i,y_j),\ (i=1,\ \dots\ ,\ n;\ j=1,\ \dots\ ,\ m)$$

The notation " :=" below means "defined to be".

To construct the SDG (Surface of the R.Q. of the **H** in the direction of the Gradient):

1. Determine the direction of the steepest drop of the seafloor at the point (x,y). This is -**DS**, where **DS** is the gradient of S at (x,y) i.e.,

$$\mathbf{DS}:=(D_xS,\ D_yS).$$

(It is the direction perpendicular to a contour line of the seafloor through the point (x,y).)

2. Determine the normalized second derivative of S at (x,y) in the gradient direction established above. This is given by the number $\boldsymbol{N}$ where:

$$\boldsymbol{N}:=v^TD^2Sv:=(v_x)^2D_{xx}S+2v_xv_yD_{xy}S+(v_y)^2D_{yy}S, \qquad (1)$$

Where the "T" superscripting a vector means "transpose", and with the second partial derivatives, $D_{xx}S$, $D_{xy}S$, $D_{yy}S$, of the function S all calculated at the point (x,y) and with

$$v:=(v_x,v_y)=\mathbf{DS}\ /\ \|\ \mathbf{DS}\ \| \qquad (2)$$

where $\|v\|^2:=v_x^2+v_y^2$, i.e., v is the normalized gradient of S at (x,y). The calculation of $\boldsymbol{N}$ is certainly easier than the calculation of the value needed in the computation of the SMC.

Note that, with v such that:

$$v=\mathbf{DS}\ /\ \|\ \mathbf{DS}\ \|,$$

equation 1 can be rewritten as:

$$\boldsymbol{N}=[(\mathbf{DS})^T\mathbf{H}(\mathbf{DS})]\ /\ [(\mathbf{DS})^T(\mathbf{DS})] \qquad (3)$$

Where **H** *the Hessian matrix of* S *is defined to be:*

$$\mathbf{H} := \begin{bmatrix} D_{XX}\,S & D_{XY}\,S \\ D_{YX}\,S & D_{YY}\,S \end{bmatrix} = \mathbf{D}^2 S$$

This shows the number $\boldsymbol{N}$ to be the R.Q. for $\mathbf{H}$ evaluated at the vector, $v=\mathbf{D}S$:

$$R_{\mathrm{H}}(v) := N = (v^{\mathrm{T}}\mathbf{H}v) \,/\, (v_{\mathrm{T}}v) \qquad (4).$$

The relationship between the SDG of Equation 4 and the SMC is as follows:

If by the word "curvature" one means nothing more than the second derivative in any particular normalized direction, then by the argument just given, the maximum curvature would be the maximum of the R.Q. of **H**, i.e., the maximum eigenvalue of **H**. In particular, if the gradient-directed second derivative is "large", then so must be the maximum "curvature". However, this maximum "curvature" may well be large in places where the gradient directed second derivative $\boldsymbol{N}$, i.e., the "curvature" in the gradient direction, is not large.

Actually, the SMC proposed by VANICEK et al (1994) as a means for determining the FCS is based on the actual curvature of the surface, Σ where:

$$\Sigma : (x,y) \rightarrow (x,\, y,\, S(x,y)\,),$$

and hence may be even further removed from the original intent of the legal definition of the FCS.

Specifically, the SMC is obtained as:

$$(x,y) \rightarrow (x,\, y,\, \max\{0, \max \boldsymbol{k}\,(x,y)\})$$

with max $\boldsymbol{k}(x,y)$ the maximum normal curvature of Σ at the point $(x,\, y,\, S(x,y))$. Elementary differential geometry applied to the surface Σ shows the normal curvature of Σ in the direction v equals the R.Q. evaluated at v. To see this let $\mathbf{H}^{\wedge}$ be the matrix of the dirst fundamental form for Σ, i.e.,:

$$\mathbf{H}^{\wedge} = \mathbf{H} \,/\, \|\, (-\mathbf{D}S,1)\, \|\,,$$

and **G** be the matrix of the first fundamental form for Σ, i.e.,:

$$\mathbf{G} = (\mathbf{D}\Sigma)^{\mathrm{T}}(\mathbf{D}\Sigma) = \begin{bmatrix} 1+(D_xS)^2 & D_xSD_yS \\ D_ySD_xS & 1+(D_yS)^2 \end{bmatrix},$$

then

$$\boldsymbol{K}(v) := R_{\mathrm{H}^{\wedge},\,\mathrm{G}}(v) := [v^{\mathrm{T}}\mathbf{H}^{\wedge}v \,/\, v^{\mathrm{T}}\mathbf{G}v] = [v^{\mathrm{T}}\mathbf{H}v : v^{\mathrm{T}}\mathbf{G}v] \,/\, \mathrm{SQRT}\,[\mathbf{D}_xS^2 + D_yS^2 + 1]. \qquad (5)$$

For the specific choice $v=\mathbf{D}S$, one has

$$\boldsymbol{K}(\mathbf{D}S) = \boldsymbol{N} : \|\, (-\mathbf{D}S,\, 1)\, \|^3,$$

hence a simple relationship between the SDG, $\boldsymbol{N}$, and the SMC, $\max_v \boldsymbol{K}(v)$. Since $\|(-\mathbf{D}S,\, 1)\,\|$ may vary widely, there may be no connection between the maxima of $\boldsymbol{N}$

and those of the SMC. This makes the SMC even more doubtful in computing the FCS. Since both approaches use second derivative information, careful smoothing of the original data is imperative to a correct location of the FCS by either method.

Since both $\mathbf{H}^{*}$ and $\mathbf{G}$ are real symmetric, the maximum normal curvature at a point is the larger of the two principle curvatures at that point, i.e., the larger of the two eigenvalves k_1, k_2 of the generalized eigenvalve problem:

$$\mathbf{H}^{*} - k\mathbf{G}.$$

Note also that $\boldsymbol{k}_1 \leq \boldsymbol{N} \leq \boldsymbol{k}_2$, hence at any point the SDG is less than or equal to the SMC.

Equivalent, max $\boldsymbol{k}$ is the larger of the two solutions of the quadratic equation.

$$\det(\mathbf{H}^{*} - k\mathbf{G}) = 0 ,$$

which, on expanding the determinant and collecting terms according to powers of $\boldsymbol{k}$, gives exactly the result of Equation 2 in VANICEK et al. (1994)(as it should be).

Table 1.- The steps of the SDG algorithm

1. Let $S(x,y)$ be an explicit mathematical spline representation obtained from discrete bathymetric rectangular array of $Z(x,y)$, which represents the seafloor.

2. Specify a rectangular grid to be used, i.e., partition to be used in the x and y vectors. Let (x,y) be any point in the grid.

3. *Compute the gradient vector* $\boldsymbol{v}$ *at* (x,y):

$$\boldsymbol{v}(x,y) \doteq \mathbf{D}S(x,y),$$

4. Compute the Hessian matrix $\mathbf{H}$ of $S(x,y)$ at the point (x,y):

$$\mathbf{H} := \begin{bmatrix} D_{XX}S & D_{yx}S \\ D_{yx}S & D_{yy}S \end{bmatrix},$$

5. Compute the Rayleigh Quotient R of the vector $\boldsymbol{v}$ at (x,y):

$$R(x,y) = [(\boldsymbol{v})^T\mathbf{H}(\boldsymbol{v})] / [(\boldsymbol{v})^T(\boldsymbol{v})].$$

(superscript T means transpose)

6. Enter the value of $R(x,y)$, obtained from the Rayleigh Quotient in the above step 5 for each point (x,y) of the grid to obtain the SDG.

Note: The location of the crest of the highest ridge on the SDG surface is the estimated FCS.

The SDG requires only x and y axes have the same coordinate units. Rescaling units only changes $\boldsymbol{N}$ in Equation 3 by a constant. The SMC requires the x, y, and z axes all have the same coordinate units. The SQRT term in computing $\boldsymbol{K}$ in the denominator of Equation 5 causes the SMC to be very sensitive to scaling.

Finally, the simple real examples that follow show that the original legal definition of the FCS does not always cover every situation. There are places where the passage from a steep descent to flattish continental rise of the seafloor can be quite gradual with no particular area of sharp change in gradient. For this situation, an alternate legal definition of the FCS seems needed.

See table 1 for a concise statement of the SDG algorithm.

EXAMPLE: THE U.S. ATLANTIC FOOT OF THE CONTINENTAL SLOPE

The procedures outlined above will be implemented on the NOAA worldwide bathymetric dataset ETOPO5 by computing the SDG and the SMC for the area outlined from longitude -76 to -69 and from latitude 33 N to 40 N. This is the U.S. Atlantic coast from Charleston, South Carolina, to just south of New York, New York. Figure 2 shows an outline of this area on a map. This 7°x7° area encloses a sizable portion of the U.S. Atlantic FCS.

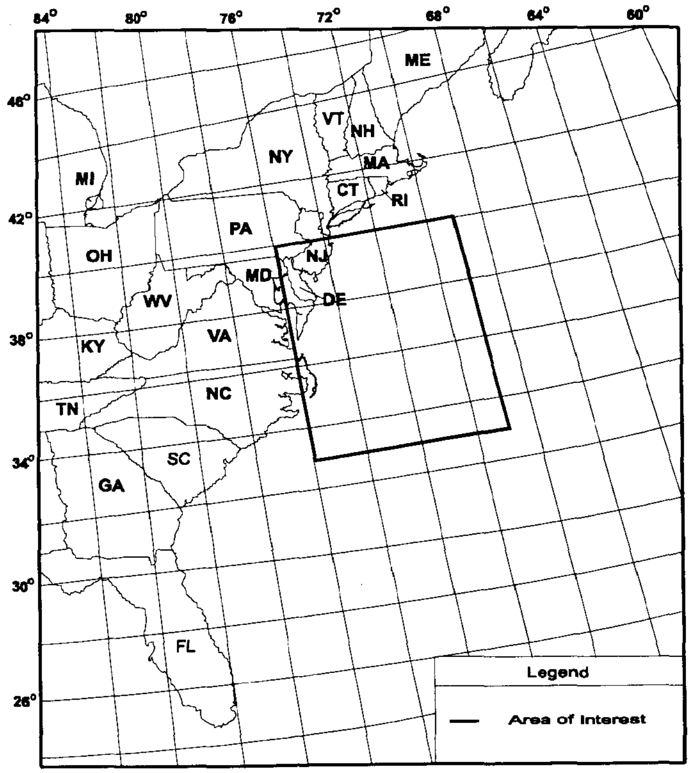

FIG. 2.- Map of study area of U.S. Atlantic coast.

Let:

$$z(x_i, y_j), (i=1, \ldots, n; j=1, \ldots, m)$$

be the ETOPO5 subsea bathymetric data given on an equally spaced 5-nautical mile rectangular grid. (Information from scattered data would have to be transformed to a rectangular grid prior to using this technique.) Figure 3 shows a 3-dimensional net display of the ETOPO5 data described in the previous paragraph and given by the 85 x 85 grid:

$$z(x_i, y_j), (i=1, \ldots, 85; j=1, \ldots, 85),$$

returned by spline-smoothing Matlab M-file TCSAPS. TCSAPS has smoothing parameters p_x and p_y, where $0 \leq p_x \leq 1$ and $0 \leq p_y \leq 1$. With p_x and p_y set to 1, there is no smoothing. With p_x and p_y set to 0, there is the most smoothing. Usually p_x and p_y are chosen such that $p=p_x=p_y$.

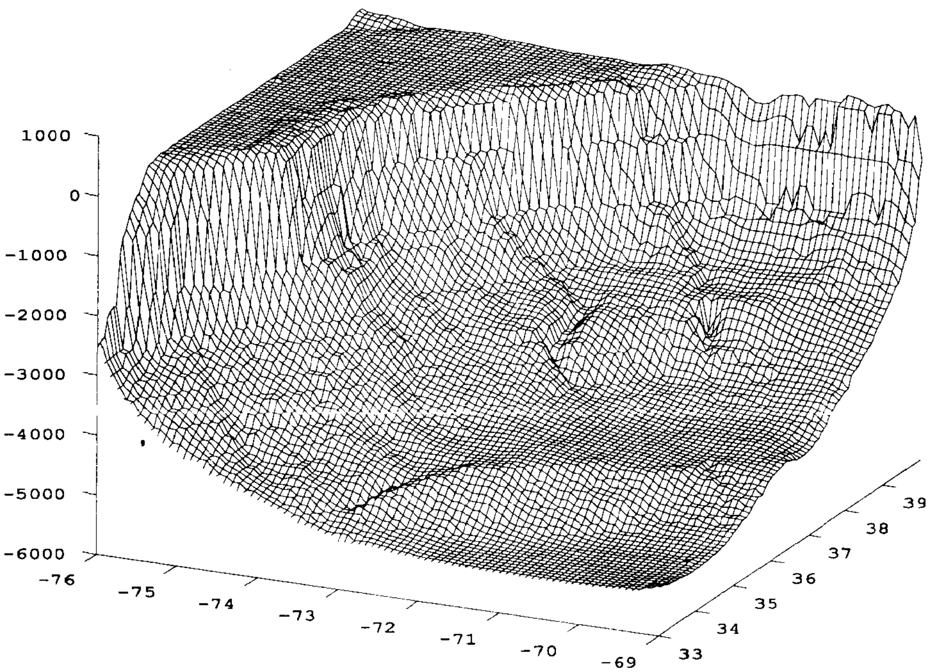

FIG. 3.- 3-D Net display of original NOAA ETOPO5 data of 7 degrees longitude by 7 degrees latitude portion of U.S. Atlantic coast ('Z=0' is sea level; 5-nautical mile grid interval (85x85 grid)) p=1.

For clarity of presentation, contour maps will be used to show the effects of data smoothing by TCSAPS for various values of the smoothing parameters $p=p_x=p_y$. The first contour map is Figure 4, which is a contoured presentation of Figure 3, the original data. This was accomplished by M-file TCSAPS with the smoothing parameters p_x and p_y such that $p=p_x=p_y=1.0$, which is no smoothing, just a $S(x,y)$ interpolation of the raw ETOPO5 data. The contour interval on this map is 500 metres . The contours presented are from -500 metres to -5000 metres below sea level. Note the angularity and sharp points on most of the contours. Also note the prominent presence of Hudson Canyon as it affects the contours located between (-70°, 36°) and (-71°, 40°).

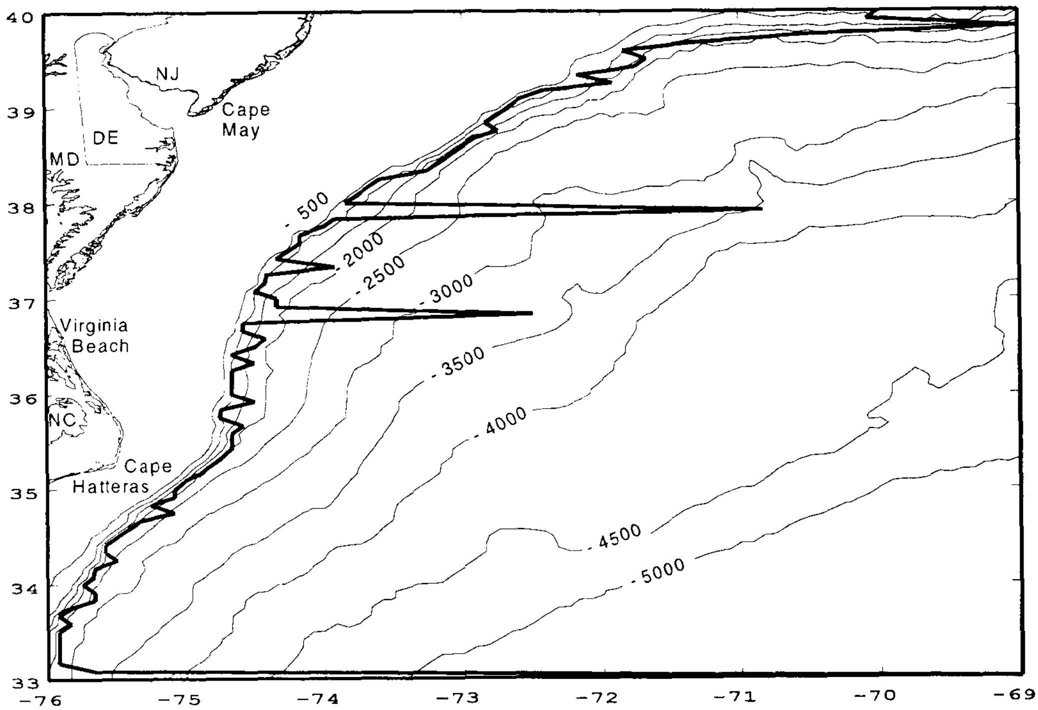

FIG. 4.- Contour map of original NOAA ETOPO5 data with smoothing parameters: $p = p_x = p_y = 1.0$ of 7 degrees longitude by 7 degrees latitude portion of U.S. Atlantic coast (contour interval = 500 metres ; 5-nautical mile grid interval l(85 x 85 grid); heavy line is a plot of the FCS as found by SDG.)

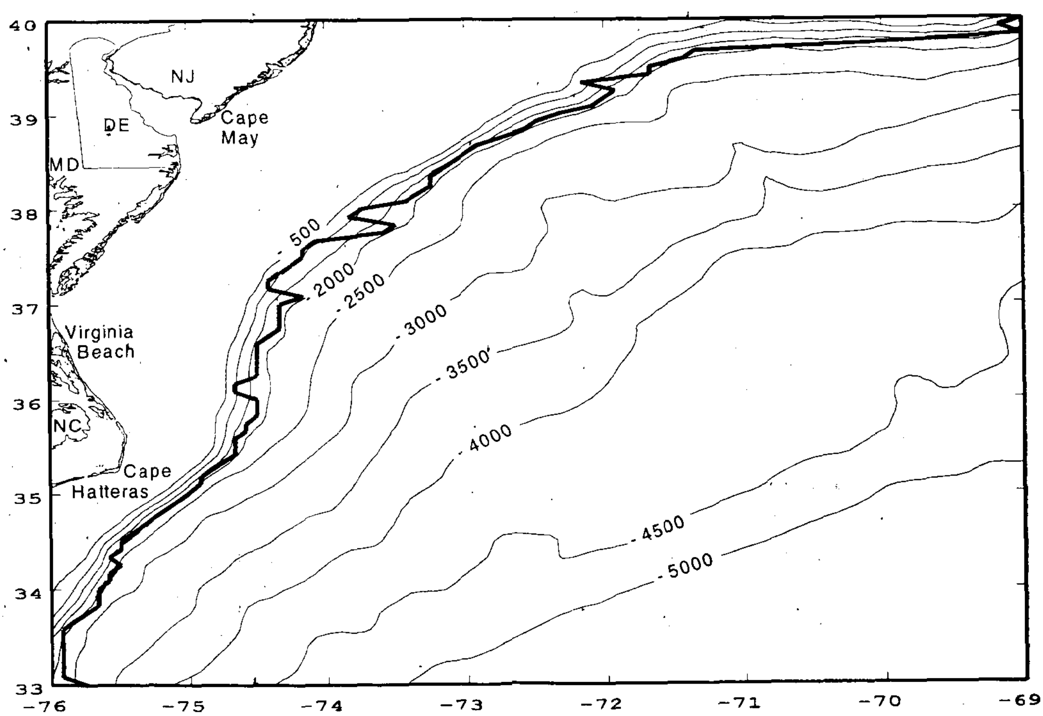

FIG. 5.- Contour map of the spline-smoothed NOAA ETOPO5 data with smoothing parameters: $p = p_x = p_y = .9999$ from 7 degrees longitude by 7 degrees latitude portion of U.S. Atlantic coast (contour interval = 500 metres ; 5-nautical mile grid interval (85 x 85 grid); heavy line is a plot of the FCS as found by SDG).

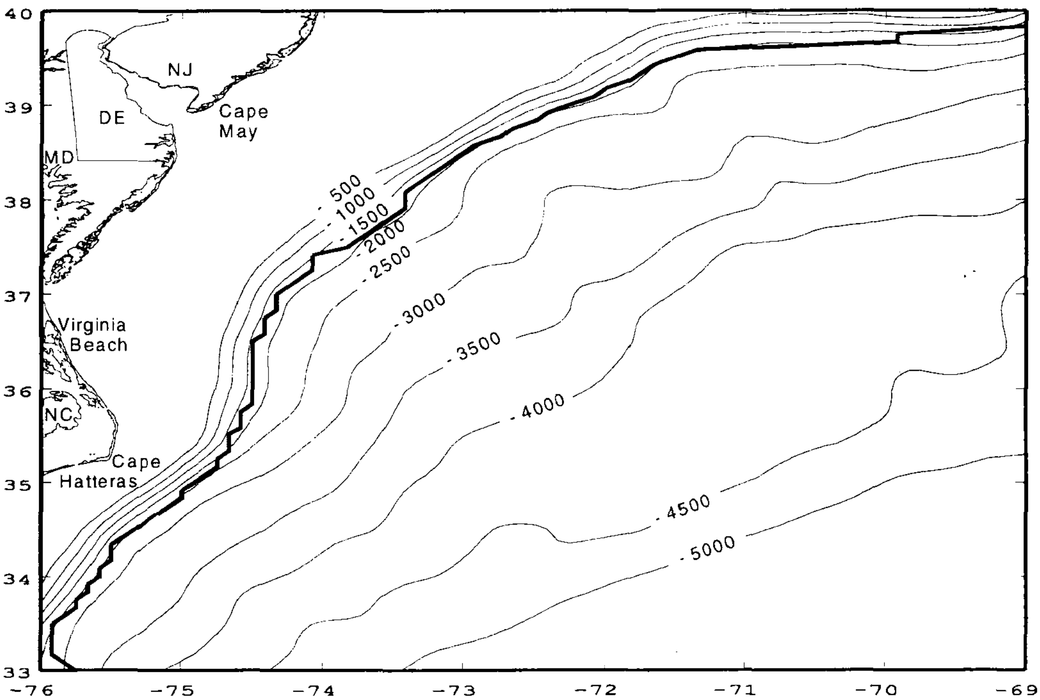

FIG. 6.- Contour map of spline-smoothed NOAA ETOPO5 data with smoothing parameters: $p = p_x = p_y = .9990$ from 7 degrees longitude by 7 degrees latitude portion of U.S. Atlantic coast (contour interval = 500 metres ; 5-nautical mile grid interval !(85 x 85 grid); heavy line is a plot of the FCS as found by SDG).

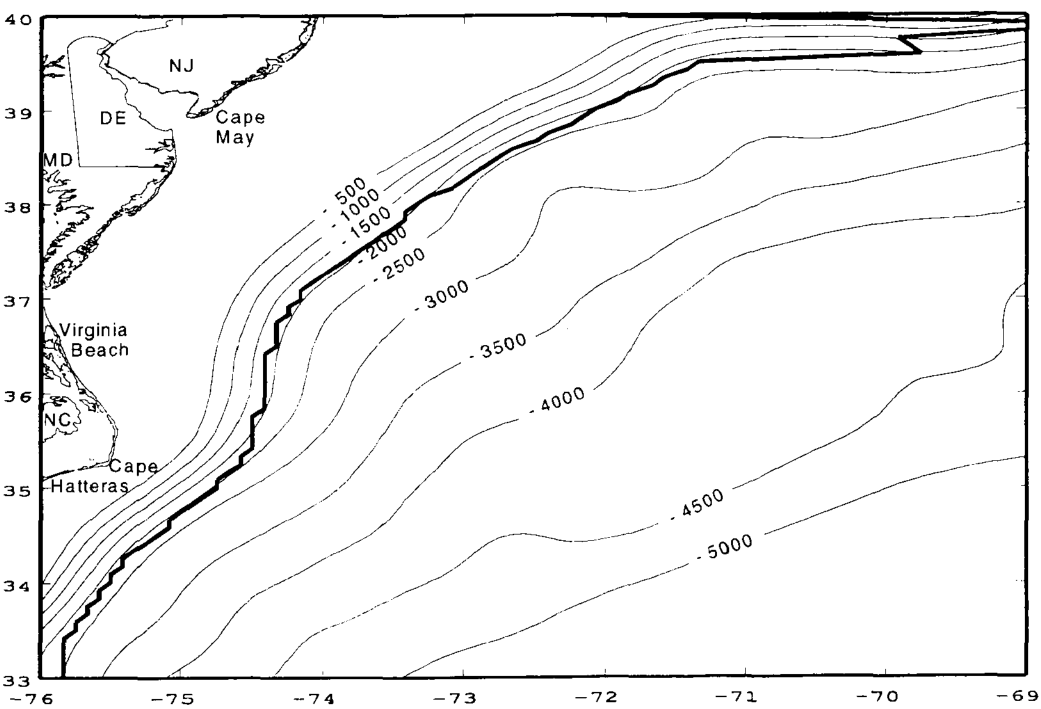

FIG. 7.- Contour map of the spline-smoothed NOAA ETOPO5 data with smoothing parameters: $p = p_x = p_y = .9900$ from 7 degrees longitude by 7 degrees latitude portion of U.S. Atlantic coast (contour interval = 500 metres ; 5-nautical mile grid interval (85 x 85 grid); heavy line is a plot of the FCS as found by SDG).

The SDG procedure was run on this raw dataset and the results plotted on the contour map of Figure 4. In this and other contour maps, the FCS is indicated by a heavy black line. This line is generated and drawn by the computer software. The FCS jumps from the -500 metre contour to the -3500 metre contour at 37° N latitude and 38° N latitude as observed in Figure 4. The FCS between Cape Hatteras, North Carolina, and Virginia Beach, Virginia, zigs and zags between the -500 metre contour and the -2000 metre contour. Notice the close spacing for the -500 metre, -1000 metre, and -1500 metre contours. This represents a steep dip or gradient.

To smooth out some noise, set p = p_x = p_y = .9999. The results are plotted in Figure 5. Note that the contours are less angular and less pointed but their spacial location is still the same as well as the relative distance between them. Also note the Hudson Canyon on the right side of the map is still well defined; hence, there has been little loss of information content by this slight degree of smoothing.

The FCS as computed by the SDG method still zigs and zags between the -500 metre contour and the -2000 metre contour between Cape Hatteras, North Carolina, and Cape May, New Jersey; however, the large spikes at 37°N and 38° N have been eliminated. More smoothing is required.

Set p = p_x = p_y = .9990 and consider the results as given in Figure 6. The contours have been smoothed further. There are no sharp points nor angular edges remaining in the contours. Note that the relative distance between the contours on this map has been preserved and is about the same as the original data when p = 1 in Figure 4. The Hudson Canyon on the right side of the map is still clearly evident. From Cape Hatteras, North Carolina, to Cape May, New Jersey, with only one exception, the FCS is contained between the -1500 metre contour and the -2000 metre contour. This is where it should be, according to the legal definition, because this is exactly where the "maximum change in the gradient at its base" occurs, i.e., after the closely spaced contours that indicate a steep slope; the foot is located by an increase in the spacing between the contours, which is where the maximum change in the gradient at the base occurs.

To introduce more smoothing, set p = p_x = p_y = .9900. The results are given in Figure 7. Now the contours are very smooth. Note the -500 metre, -1000 metre, and -1500 metre contours in Figure 6 are close together and -2000 metre, -2500 metre, -3000 metre contours begin to widen out. In Figure 7, the -500, -1000, (-1500, and -2000 contours are about equally spaced. Note also the Hudson Canyon on the right side of the map is essentially nonexistent. Although this is a map with nice smooth contours, it has lost much of the information content of the original data in this last smoothing increment, i.e., there has been too much smoothing of the data by setting p = p_x = p_y = .9900. So p = p_x = p_y = .9990 is optimal if we increment the smoothing parameter by dropping off a 9 at each step. This is the data set that will produce the best results for a two derivative procedure with p = .9990.

An equation for p is given on page 2-13 by de BOOR (1992) to be:

$$p = 1 / [1+(h^3/6)], \tag{6}$$

where h is the grid interval. Using de BOOR's Equation 8, one finds that when h=1/12, then p=.9999. For this dataset and the objectives of this report, Figure 5 shows that this is not enough smoothing. The results from this report would suggest replacing the constant 6 in Equation 6 with a constant of .6; thus, for h=1/12, then p=.9990, which yields the optimum value of *p* found for this dataset.

In the previous figures, the line representing the FCS was plotted on the contour maps without explaining how the lines representing the FCS were obtained. The intermediate steps that were used to obtain results presented will now be given. Figure 8 is a 3-dimensional net display of the surface generated by the SDG procedure as given by the SDG algorithm in table 1 above. Note the crest of the highest ridge on the surface locates the FCS and runs from the lower left-hand corner to the upper right-hand corner of the display. Also, note the small ridge just to the left of the big FCS ridge. It represents the place in the data where it was close to the coast line, and from there to the coast line all the *z* values were arbitrarily set to a value of -10 metres in the original dataset. The smaller features to the right of the FCS ridge are of interest. These trends should be examined in more detail by oceanographers to see what they represent.

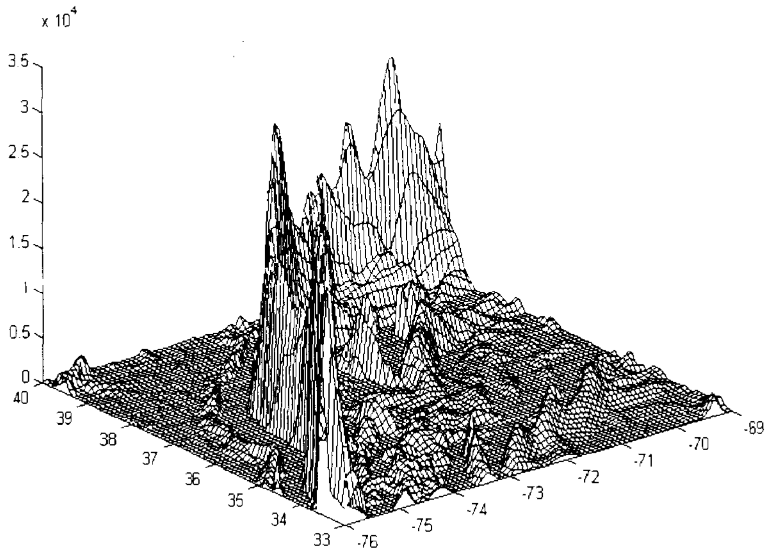

FIG. 8.- 3-D Net display of the SDG of spline-smoothed NOAA ETOPO5 data with $p + p_x + p_y = .9990$ from 7 degrees longitude by 7 degrees latitude portion of the U.S. Atlantic coast (5-nautical mile grid interval (85x85 grid); FCS is located by the peak of the highest ridge)

The FCS line plotted on all the previous contour maps was obtained by contouring the SDG of Figure 8. This is displayed in Figure 9. The line indicating the FCS displayed in Figure 9 was found by the program keeping a record of where the cells of the grid obtained the largest value and connecting a line between those cells as the computation progressed.

One now shows that the 85x85 grids can be partitioned to a finer grid. The smoothing parameter p = .9990 eliminated much of the noise and still maintained the information content of the ETOPO5 *z* values. These data are on a 5-nautical mile grid interval. At this point, because the bi-cubic smoothing spline is a closed mathematical function, it could be sampled to as fine a grid as desired. Because this dataset was already quite large with an 85x85 grid, the grid was only refined once in this example by adding twice the number of equally spaced points in the *x* vector and the *y* vector to define the grid to be 170x170 with a 2.5-nautical mile grid interval.

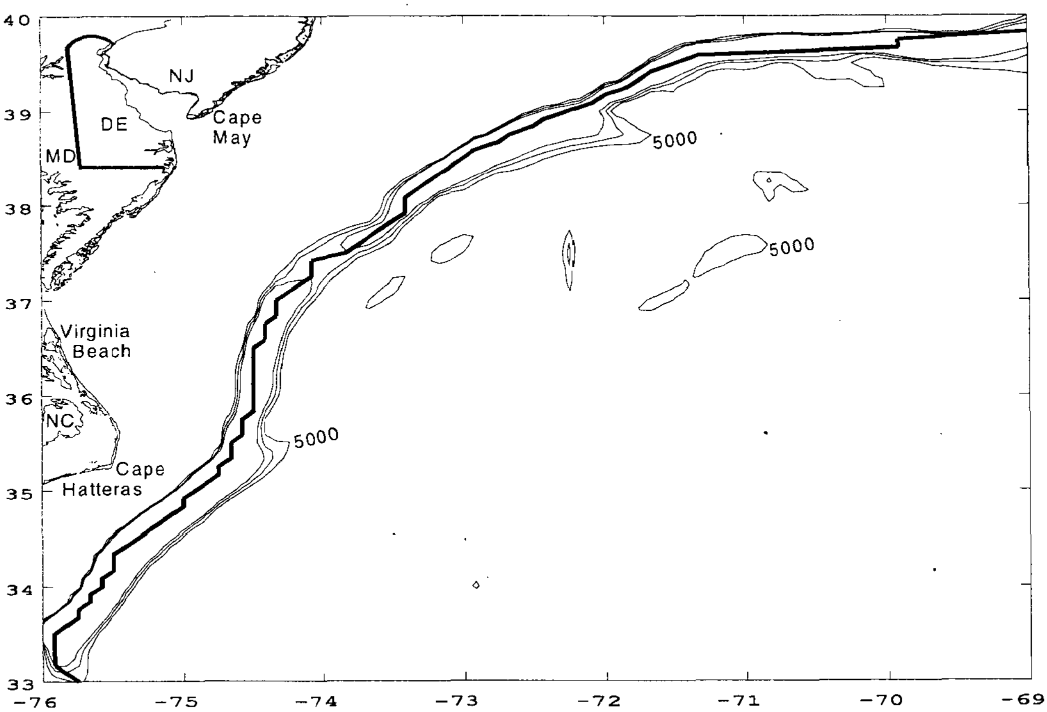

FIG. 9.- Contour map of the SDG using the spline-smoothing NOAA ETOPO5 data with smoothing parameters: $p = p_x = p_y = .9990$, 7 degrees longitude by 7 degrees latitude portion of U.S. Atlantic coast (FCS is obtained from the crest of the highest ridge; 5-nautical mile grid interval (85x85 grid); heavy line is a plot of the FCS as found by SDG).

Note in Figure 3 the noisy 3-D net display of *S*(*x*,*y*) was given for the original ETOPO5 dataset with smoothing parameter p = 1.0. Figure 4 showed what happened when this dataset was contoured and the SDG computed with this *S*(*x*,*y*) as input. This gave a very poor location of the FCS at a grid interval of 5 nautical miles. Figure 10 is a 3-D net display of *S*(*x*,*y*) with an optimal smoothing parameter of p = .9990. The data have been smoothed and the FCS plotted at a grid interval of 2.5 nautical miles. It misses the FCS at some places; but it is locating it at most of them.

Figure 11 is a contour map of Figure 10. The FCS is located in map view by latitude and longitude to the accuracy of the 2.5 nautical mile grid. The bi-cubic spline-smoothing representation *S*(*x*,*y*) with an optimal smoothing parameter does an excellent job of eliminating most of the noise and maintaining the data integrity and information content.

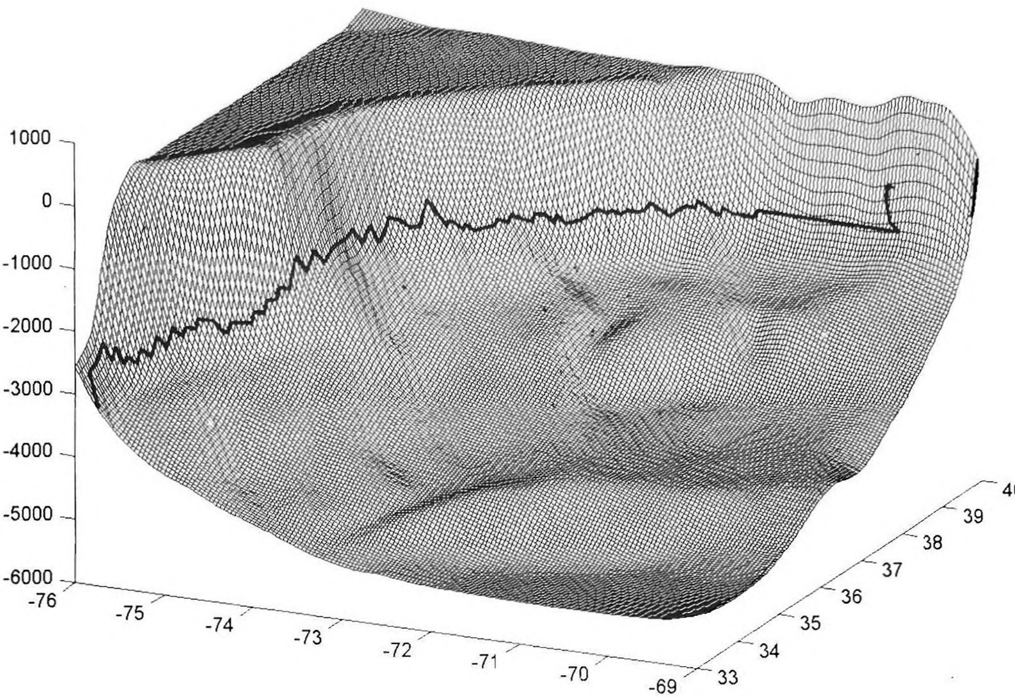

FIG. 10.- 3-D Net display of the spline-smoothed NOAA ETOPO5 data with smoothing parameters: $p = p_x = p_y = .9990$ from 7 degrees longitude by 7 degrees latitude portion of U.S. Atlantic coast (Z=0' is sea level; 2.5-nautical mile grid interval (170x170 grid); heave line is a plot of the FCS as found by SDG).

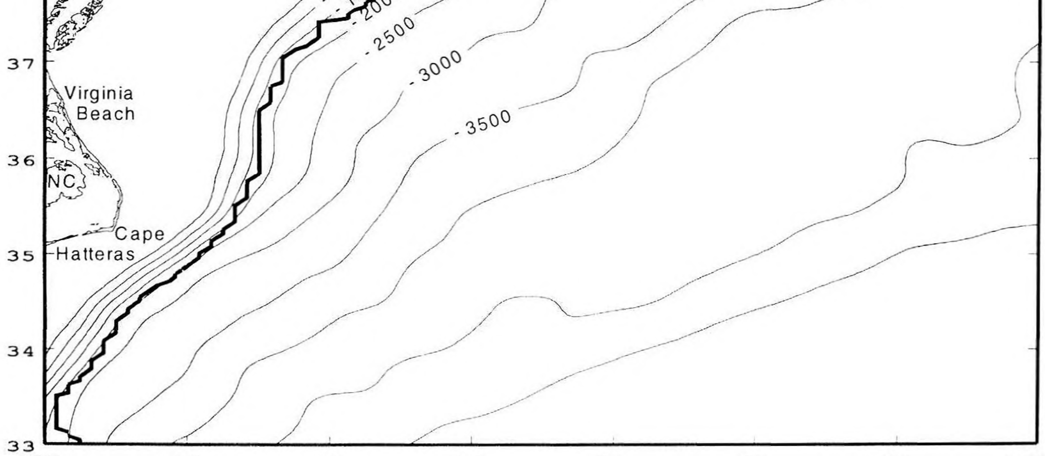

FIG. 11.- Contour map of the spline-smoothed NOAA ETOPO5 data with smoothing parameters: $p = p_x = p_y = .9990$ from 7 degrees longitude by 7 degrees latitude portion of U.S. Atlantic coast (contour interval = 500 metres ; 2.5-nautical mile grid interval (170 x 170 grid); heavy line is the FCS as found by SDG

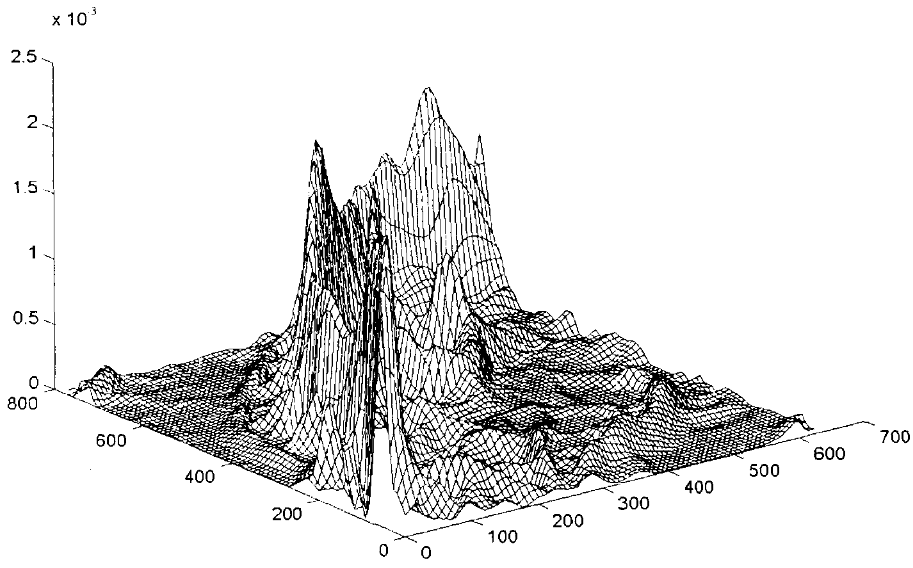

FIG. 12.- 3-D Net display of the SMC; x, y and z axes are in kilometres using the spline-smoothed NOAA ETOPO5 data with smoothing parameters $p_x = 7.00728 \times 10^{-4}$, $p_y = 3.650 \times 10^{-4}$ 7 degrees longitude by 7 degrees latitude portion of U.S. Atlantic coast (28.135 kilometre grid interval *x*-axis; 35.000-kilometre grid interval *y*-axis (85x85 grid); FCS is the peak of the highest ridge) .

FIG. 13.- Contour map of the SMC; *x*, *y* and z axes in kilometres with smoothing parameters: $p_x = 7.00728 \times 10^{-4}$, $p_y = 3.650 \times 10^{-4}$ 7 degrees longitude by 7 degrees latitude portion of U.S. Atlantic coast (contour values in kilometres as labeled; 28.135-kilometre grid interval *x*-axis; 35.000-kilometre grid interval *y*-axis (85 x 85 grid); heavy line is the FCS as found by SMC).

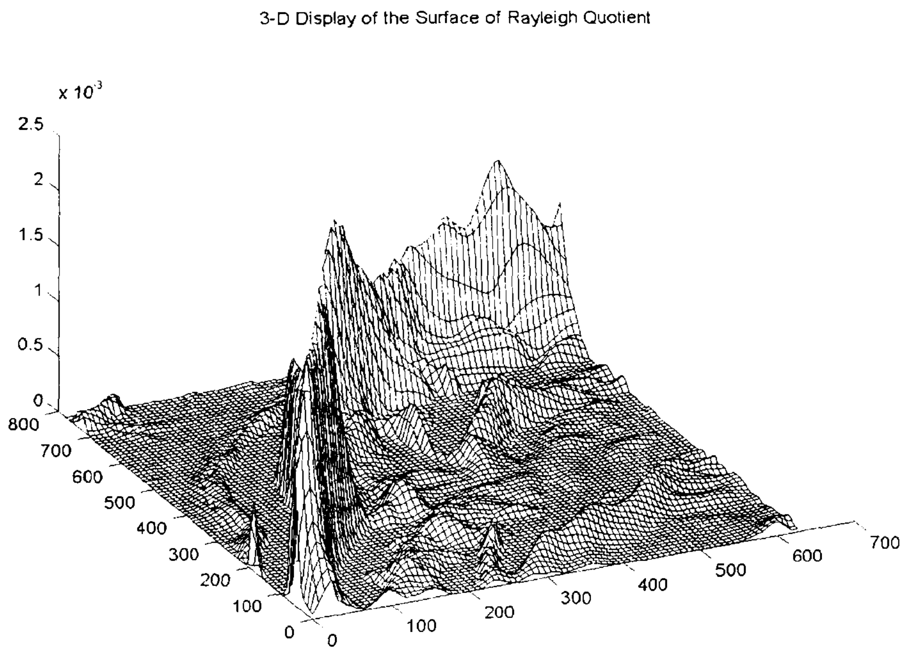

FIG. 14.- 3-D Net display of the SDG, *x*, *y*, and *z* axes are in kilometres using the spline-smoothed NOAA ETOPO5 data with smoothing parameters; $p_x = 7.00728 \times 10^{-4}$, $p_y = 3.650 \times 10^{-4}$ 7 degrees longitude by 7 degrees latitude portion of U.S. Atlantic coast (28.135-kilometer grid interval *x*-axis; 35.000-kilometre grid interval *y*-axis (85x85 grid); FCS is the peak of the highest ridge)

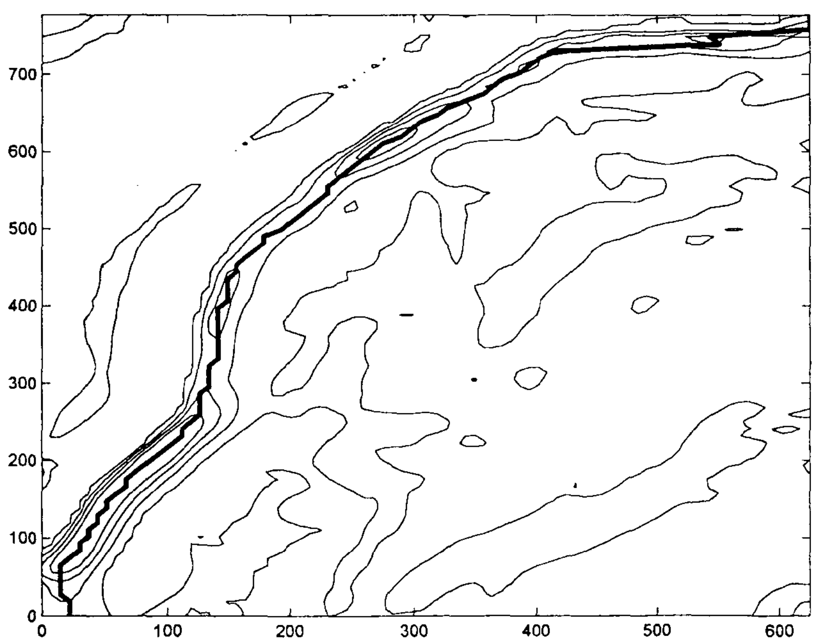

FIG. 15.- Contour map of the SDG, *x*, *y*, and *z* axes in kilometres with smoothing parameters: $p_x = 7.00728 \times 10^{-4}$, $py = 3.650 \times 10^{-4}$ 7 degrees longitude by 7 degrees latitude portion of U.S. Atlantic coast (contour values in kilometres as labeled; 28.135-kilometer grid interval *x*-axis; 35.000-kilometer grid interval *y*-axis (85x85 grid); heavy line is a plot of the FCS as found by SDG).

This ETOPO5 dataset was chosen because it covered most of the coastal areas of the world. Working with the noisy original ETOPO5 data would have produced poor results with any approach attempting to compute a surface that has to be differentiated twice, because taking the derivative of any real dataset always increases the noise level.

The SDG approach described above will be contrasted for this example with the SMC as outlined by VANICEK et al. (1994). The SMC was computed using Equation 1-5 and 11 of their report. The results of running the SMC on this dataset as it originally was given in *x*: degrees, *y*: degrees and *z*: metres are not presented, because it gave no reasonable location of the FCS. In the Introduction, it was explained why this would happen without scaling. In order to obtain meaningful results with the SMC, the units of measurement on all axes needed to be changed to the same *units. The input used had a smoothing parameter of* p = .9990, which previously was found to be the optimal smoothing parameter for this dataset. With the data grid scaled so that the grid units are kilometres on the *x, y,* and *z* axes, the SMC's peak of its highest ridge locates the FCS. The scaled SMC map of the FCS then needs to be scaled again to return the map to the proper aspect ratio of the original data. As pointed out in the Introduction, the SDG approach requires only the *x* and *y* axes have the same units and requires no scaling of the ETOPO5 data to obtain the FCS. To accommodate the SMC, the data are now scaled to compare the location of the FCS using the SMC and the SDG.

To scale the *x* and *y* axes of the dataset, the degrees were concerted to kilometres. For latitude, *y*, 111 kilometres for each degree was used; hence for latitudes 33° to 40° the kilometres ranged from 0 to 777 on the *y*-axis. On the *y*-axis the scaling factor is $r_y = 111$. *For longitudes -76° to -69°*, x, cos(36.5°) x 777 = 625; hence the scale was from 0 to 625 on the *x*-axis. On the *x*-axis the scaling factor is 89.228. The 36.5° is the mean of 40 and 33 degrees. Both axes are in kilometres.

The SMC did place the FCS in essentially the same locations as the SDG after the scaling of the *x, y* and *z* axes to kilometres. The 3-D net displays of the scaled axes of the SMC and SDG are given in Figures 12 and 14, respectively. Note the sharp rise from 0 of the ridge that represents the FCS as found by the SDG in Figure 14. To the west of the FCS is a smaller, secondary ridge of essentially the same orientation. The smaller ridge is where a -10 metres below sea level truncation of the data occurs. In Figure 12, the SMC does not show this as a distinct ridge; but it has only one wide ridge with no clear break between the two ridges. Figures 13 and 15 are the contour maps of the SMC and SDG, respectively. Note that the SMC in Figure 13 and the SDG in Figure 15 the highest ridge (FCS) run from the lower left corner to the upper right corner on both. But the ridge on the SDG in Figure 15 is better defined. In general, the features in the SMC in Figure 13 are not as well defined, nor distinct; rather, they run together. The SDG obtained the same location of the FCS at all scales. Compare Figures 9 and 15 which have different scales. There is no figure for the FCS by SMC because it cannot compute it at the original scale of the data.

When scaling in figures with scaling factors, r_x and r_y, the following Equations are recommended for the transformed coordinates to determine the best smoothing parameters p_x and p_y:

$$p_x = p_{xo} / [p_{xo} + (1 - p_{xo}\, r_x^3],$$

and

$$p_y = p_{yo} / [p_{yo} + (1 - p_{yo}\, r_x^3],$$

With $p = p_{xo} = p_{yo} = .999$ and $r_x = 89.228$ and $r_y = 111$, the optimum smoothing parameters are found to be $p_x = 1.4\text{x } 10^{-3}$ and $p_y = 7.2993 \times 10^{-4}$. Actually, in the transformed space, the best results were obtained by scaling p_x and p_y by 0.5 obtaining $p_x = 7.0078 \times 10^{-4}$ and $p_y = 3.650 \times 10^{-4}$.

For this ETOPO5 dataset and the processing used in the report, the SDG has features that are better defined and more distinct than those same features as outlined by the SMC. It was shown in theory, in the paragraph following Equation 4, that, "this maximum curvature may well be large in places where the gradient directed second derivative, *N*, i.e., the curvature in the gradient direction is not large." This property may afford the SDG more flexibility and freedom than the SMC and thus allow it to be a more sensitive indicator of the FCS. More data need to be run to verify this conjecture.

A numerical verification of this theoretical observation in the paragraph following Equation 4 about the relative heights of the two surfaces will now be given. The surface Δ = SMC - SDG was computed as the difference of the two surfaces. Every value of the surface Δ was found to be zero or positive. This is clearly shown in Figure 16 by the 3-D net plot of the surface Δ.

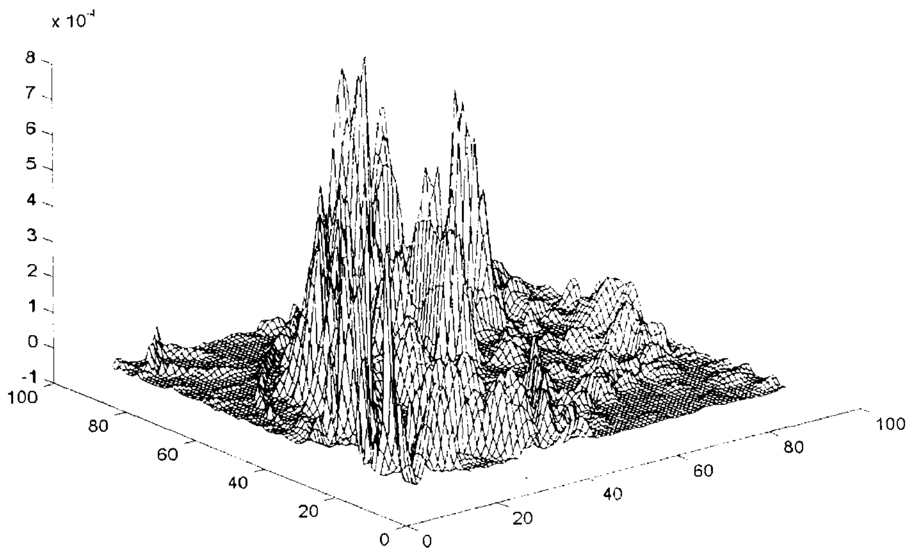

FIG. 16.- 3-D Net display of Δ = SMC - SDG, *x*, *y*, and z axes are in kilometres using the spline-smoothed NOAA ETOPO5 data with smoothing parameters; $p_x = 7.00728 \times 10^{-4}$, $p_y = 3.650 \times 10^{-4}$ 7 degrees longitude by 7 degrees latitude portion of U.S. Atlantic coast (28.135-kilometre mile grid interval *x*-axis; 35.000-kilometre grid interval *y*-axis (85x85 grid)).

This approach of bi-cubic smoothing and data representation should make NOAA's ETOPO5 worldwide database of interest and value in many areas of oceanography. Previous to the smoothing methods of this report it has been too noisy to be of much interest.

SUMMARY AND CONCLUSIONS

1. The preprocessing of the NOAA's ETOPO5 worldwide bathymetric dataset by bi-cubic spline smoothing provides an explicit mathematical function $S(x,y)$ that reduces the noise and retains most of the original information content of the data. This representation of the ETOPO5 dataset should provide valuable information about the world's seafloors to many areas of oceanography.

This data smoothing is requisite to obtaining good results in constructing any second derivative surface. Noisy data will always give poor results using gradient methods, because taking derivatives introduces angularity and magnifies the noise. There is a limit to the amount of smoothing that should be done. An equation is given that provides a smoothing parameter p as a function of the grid interval. This parameter p should not destroy the original information content of the data. The constant required in the equation for an optimal p will vary depending on the dataset. The explicit, mathematical bi-cubic spline function allows the display of the data at a finer grid. These techniques can be implemented on any digital dataset having a rectangular grid.

2. The SDG approach is an accurate mathematical modeling of the legal definition of the FCS. The FCS is given by tracing the peak of the highest ridge of the SDG, $\Sigma : (x,y,S(x,y))$. The SDG is obtained by computing the gradient at any point (x,y) of the grid, then computing the R.Q. of the $\mathbf{H}$ matrix of $S(x,y)$ in the direction of the gradient at (x,y). The SDG contains other smaller but distinct ridges representing deep-water features seaward of the FCS. These should be examined by oceanographers in further study. See Figure 8.

3. The SDG on the EOPTO5 dataset was computationally stable and does not require special scaling of the data to obtain accurate results.

4. The SDG will not find the FCS in all cases. In particular, when a cross-section perpendicular to the contours of the continental shelf is the arc of a circle, the legal definition will not yield the FCS. Because of the uniform gradient in this case, there is no maximum gradient at the base. In such cases, an alternative definition to the current definition of the FCS must be used. When the FCS is formed under normal sedimentation conditions, this situation will be rare.

5. The theoretical mathematical relationship between the SDG and the SMC is presented in this report. The SDG uses the R.Q. of the $\mathbf{H}$ of S in the direction of the gradient to find the FCS. It has fewer mathematical steps than computing the SMC.

The SDG requires only the x and y axis to have the same units to locate the FCS accurately. The SMC required scaling the three axes to the same units to find the FCS. It must then be scaled again to restore the proper aspect ratio for the proper location of the FCS. Equations are given for the equivalent optimal smoothing parameters after scaling.

The SDG is always less then or equal to the SMC at each (x,y). This was shown in the theory on page 56 and verified in the example.

6. With new technology allowing deeper drilling and mining, coastal countries are going to be increasingly more interested in the accurate location of their FCS when it is past the EEZ.

7. It is hoped that this report will show how to utilize the information content of the NOAA ETOPO5 dataset in many areas of oceanography and allow many countries to use it to compute their FCS with it.

Acknowledgments

*The software used for these computations is a Math Works, Inc., software package called Matlab. The programs are written in the Matlab script files called M-files. The Matlab Spline Toolbox, written by Professor Carl de BOOR, is used to determine $S(x,y)$ in the spline smoothing computations. He also derived the mathematical theory. I am most grateful to Professor de BOOR for the many hours of consultation, ideas, and M-files given to me that made this effort possible. I am grateful to Professor Larry SCHUMAKER for first introducing me to splines and reading this report. Walter JOHNSON and Charles MARSHALL supplied the NOAA ETOPO5 dataset and valuable consultation. I would like to express my appreciation to Professor Petr VANICEK and Dr. Ziqiang OU for introducing me to this area of research and suggesting improvements to this report. The support and encouragement of Michael HUNT, George DELLAGIARINO, John PADAN, Roger AMATO, George CARPENTER, Jackie DURHAM, Marianna FEAGANS, and James A. BENNETT at MMS in this work were much appreciated. For all this help, I am most grateful.

The use of Matlab software in the computations and obtaining the displays in this report does not constitute an MMS or U.S. Government endorsement of the product.

References

de BOOR, Carl, 1978, A Practical Guide to Splines, Applied Mathematical Sciences No. 27, Springer-Verlag, New York, New York.

de BOOR, Carl, 1992, Spline Toolbox User's Guide, The Math Works, Inc.

OU, Ziqiang and VANICEK, Petr, 1995, Automatic Tracing of the Foot of the Continental Slope, Department of Geodesy and Geomagnetics Engineering, University of New Brunswick, PO Box 4400, Fredericton, N.B., Canada EBB A.

SCHUMAKER, Larry L., 1981, Spline Functions/ Basic Theory, John Wiley, New York, New York.

United Nations Publications: Law of the Sea: Definition of the Continental Shelf, Sales No. E.93.V.16, ISBN 92-1-133454-3.

VANICEK, Petr, WELLS, David E., and HOU, Tianhang, 1994, Determination of the Foot of the Continental Slope, Contract Report, Geological Survey of Canada, Atlantic Geoscience Centre, Bedford Institute of Oceanography, Dartmouth, N.S. B3A 4A2, SSC No. OSC93-3-R207/01-OSC.

APPENDIX A: BRIEF DISCUSSION OF SPLINE FUNCTIONS

For those unfamiliar with splines, this brief discussion of spline functions is presented because they are used in this report to smooth the raw bathymetric data. Splines are used physically by draftspersons and in ship fairing to obtain smooth curves. These splines are long plastic or wooden strips that have weights placed on them at the points the strip is to bend. The function, which is the mathematical analog of the physical spline, is also called a spline. The cubic spline is the one of interest in this report. The mathematical analog of the physical weight is called a knot. The plastic or wooden strips are represented by the mathematical splines as cubic polynomials joined together at the knots. At the breakpoints where the cubic polynomial pieces meet, they are constructed to have two continuous derivatives. To construct the second derivative surface of this report, we need to be able to take two derivatives of a function and still have a non-constant function to describe the resulting surface. The cubic spline is just what is needed.

A cubic polynomial P(x) is of the form:

$$\mathrm{P}(x) = Ax^3 + Bx^2 + Cx + D,$$

where, A, B, C, and D are real numbers. The graph of a univariate cubic polynomial P(x) is given in Figure A1.1.

A univariate cubic spline $S(x)$ is a sequence of cubic polynomials connected at the breakpoints so that the first and second derivatives match at the breakpoints. This is indicated schematically in Figure A1.B.

A univariate cubic spline can be written as:

$$S(x) = \sum_{i=1}^{n} A_i \mathrm{P}_i(x), \tag{A1}$$

where each $\mathrm{P}_i(x)$ is a cubic basis spline nonzero on $[x_i, x_{i+4}]$, where $\{x_i : i = 1, \ldots, \mathrm{n}\}$ are the knots, and the $\{A_i : i = 1, \ldots, \mathrm{n}\}$ are the real coefficients.

A bivariate or bi-cubic spline $S(x,y)$ is made up of patches of bi-cubic polynomials whose values and partial derivatives match along and parallel to the x and y axis. A schematic display of a bi-cubic spline is shown in figure A.1.C. The general form of a bi-cubic spline is:

$$S(x,y) = \sum_{i=1}^{n_x}\sum_{j=1}^{n_y} A_{i,j} P_i(x) P_j(y), \tag{A2}$$

where $\{A_{i,j} : i=1, , \mathrm{n}_x, j = 1, \ldots, \mathrm{n}_y\}$ are the n_x times n_y real number coefficients and $[\mathrm{P}_i(x): i=1, \ldots, \mathrm{n}_x\}$ are the n_x univariate basis splines in the x direction with knots at $\{x_i : i=1, \ldots, \mathrm{n}_x\}$ and $\{\mathrm{P}_j(y) : j=1, \ldots, \mathrm{n}_y\}$ are the n_y univariate basis splines in the direction with knots $\{y_j : j=1, \ldots, \mathrm{n}_y\}$, as mentioned above in the univariate case.

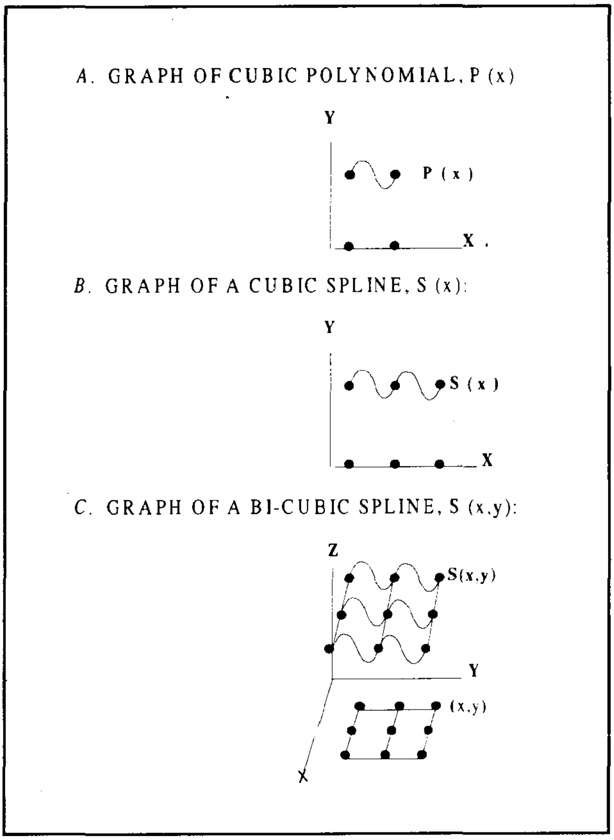

FIG. A1.- Graphs of (*A*) cubic polynomial P(*x*), (*B*) cubic spline *S*(*x*), and (*C*) bi-cubic spline *S*(*x*,*y*).

In one dimension, n data points $\{(x_i,\ y_i= \ :\ i=1,,\ n)\}$ can readily be interpolated by a unique polynomial of degree less than n by one of several polynomial interpolating procedures. The problem is that these higher order polynomials (n>9) are under such tension that in between the data points they can assume erratic values, which make polynomials of little practical use over the entire domain of definition. The Matlab M-files (script files) of the Spline Toolbox compute the coefficients $\{A_{i,j} : i=1, \ldots, n_x, j=1, \ldots, n_y\}$ for equation A2. Splines are used to represent the data rather than polynomials because splines are so much more flexible and supple than polynomials and, hence give a representation that honors the information content of the data better. They also yield better results between the original data points where the data are interpolated. The basis functions for polynomials have infinite support; hence, remote data unrelated to local data can cause poor representation locally. Because of local support of the spline basis functions, the local spline representation of the data is not distorted by remote data. The bivariate spline smoothing of this bathymetric data yields a function $S(x,y)$ of the data at the point (x,y). The bi-cubic spline function $S(x,y)$ of Equation A2 is used to represent the smoothed data, because it is the spline of lowest degree that has the continuous second partial derivatives that are required in the computation of the SDG. There is assurance that the two continuous partial derivatives exist at any point (x,y) in the domain of the function $S(x,y)$, because they exist for each of the univariate cubic spline function components from which $S(x,y)$ is constructed; hence, $S(x,y)$ has two continuous derivatives over its domain of definition:

$$[X_1, Xn_x] \times [Y_1, Yn_y].$$

The Matlab M-file, "CSAPS", is a univariate spline-smoothing program. It has a parameter p, where $0 \leq p \leq 1$. With p set to 0, the algorithm gives a least squares spline approximation to the data. With p set to 1, the algorithm gives a cubic spline interpolation of the data. When $0 < p < 1$, the representation is a weighted mixture of the two options ($p \approx 1$).

The 2-dimensional version of CSAPS is called TCSAPS. It provides the bi-cubic spline-smoothed representation of the data $\{ Z(x_i, y_j)$: $i=1, \ldots, n_x, j=1, \ldots, n_y\}$. It has two smoothing parameters p_x and p_y, $0 \leq p_x \leq 1$ and, $0 \leq p_x \leq 1$.

With p_x and p_y set to 0, TCSAPS returns a least squares linear spline approximation to the data. With p_x and p_y set to 1, TCSAP returns cubic spline interpolation of the data. Clearly, p_x and p_y can assume any value between 0 and 1, yielding a weighted mixture of least squares approximation and interpolation in the x and y direction, respectively. In the labeling of the figures, the value of the smoothing parameters p_x and p_y, will be indicated by the letter p. It is understood that $p = p_x = p_y$ in the original coordinate system.

Because we need a bi-cubic spline function with two partial derivatives in x and y direction, smoothing parameters p will be chosen close to 1.0, i.e., .9999, .9990, .9900. When the coordinate system is transformed by scaling, the proper p_x and p_y in the transformed coordinate system is given by equations (see p. 13) a function of the scaling parameters in the x and y direction.

Spline have proved to be most useful in smoothing and representing general statistical data in various fields where the raw data are noisy. For application to general statistical data, see BENNETT (1972). For application to remote sensing, pattern recognition, and image processing of digital satellite data, see BENNETT (1974) and BENNETT et al. (1974). For a more detailed presentation of spline functions, see CHENEY and KINCAID (1985); For a more theoretical discussion of splines, see SCHUMAKER (1981) and de BOOR (1978, chapter 17).

References for Appendix A

BENNETT, John O., 1972, Spline Smoothing of Histograms Using B-Splines, Rice University Computer Center, ICSA 275-025-022; Also NSAS Technical Report, NASA-CR-133987.

BENNETT, John O., 1974, Estimation of Multivariate Probability Density Functions Using B-splines, Ph. D. Dissertation, Rice University.

BENNETT, John O., de FIGURUEIREDO, Rui, P., and THOMPSON, James R., 1974, Classification by Means of B-Spline Potential Functions with Application to Remote Sensing, Invited Paper presented by J.O. Bennett at IEEE Sixth Southeastern Symposium on System Theory. Baton Rouge, Louisiana, Published in Symposium Proceedings, March 1974.

CHENEY, Ward and KINCAID, David, 1985, Numerical Mathematics and Computing, Brooks/Cole, Monterey, *California*.

DE BOOR, Carl, 1978, A Practical Guide to Splines, Applied Mathematics Sciences No. 27, Springer-Verlag, New York, New York.

DE BOOR, *Carl*, 1992, *Spline Toolbox User's Guide, The Math Works, Inc.*

SCHUMAKER, Larry L., 1981, Spline Functions: Basic Theory, John Wiley, New York, New York.

APPENDIX B: AREA OF U.S. ATLANTIC COAST WHERE FCS IS SEAWARD OF THE 200-NAUTICAL MILE EEZ

In the first area of interest presented in Figure 2 to 16 in the body of this report, the 200-nautical mile EEZ was always seaward of the FCS; therefore, the EEZ was not plotted on the maps in these figures.

We now consider the area of the U.S. Atlantic coast from -81° to -71° longitude and from 29° N to 39°N latitude. The location of this area is outlined in Figure B1. In a portion of this area off the coast of Florida, the location of the FCS (FCS plus 60 miles) is seaward of the 200-nautical mile EEZ.

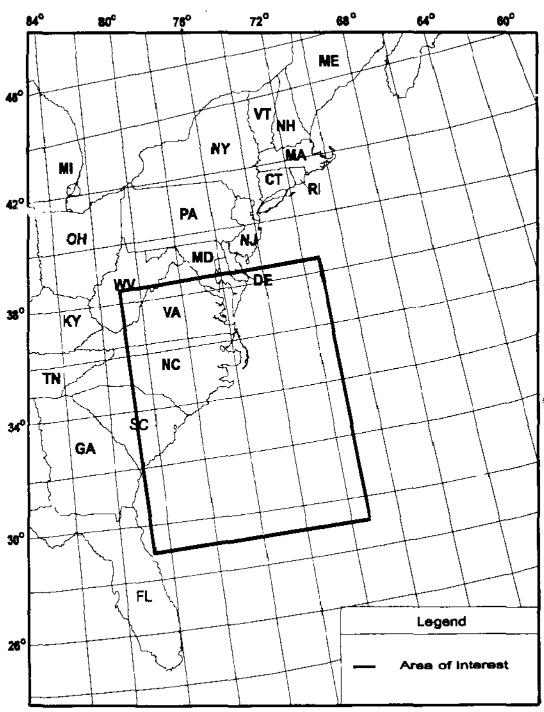

FIG. B1.- Map of U.S. Atlantic coast, Appendix B area of interest.

Figure B2 shows a more detailed contour map of this area. Article 76, item 4(a), ii: of the Law of the Sea (LOS) extends the edge of the continental margin of the United States an additional 60 nautical miles seaward of the FCS. The FCS is indicated by the heavy black line. The FCS plus 60 nautical miles is indicated by the dashed heavy line. The 200-nautical mile EEZ is located on the map by the line overwritten with circles. The area of the seafloor between the EEZ and the "FCS plus 60 nautical miles", which represents the additional mineral rights acquired by the United States by virtue of the LOS, is indicated by the shaded region.

Note that this area represents a sizable increase in the mineral rights of the United States, under the LOS.

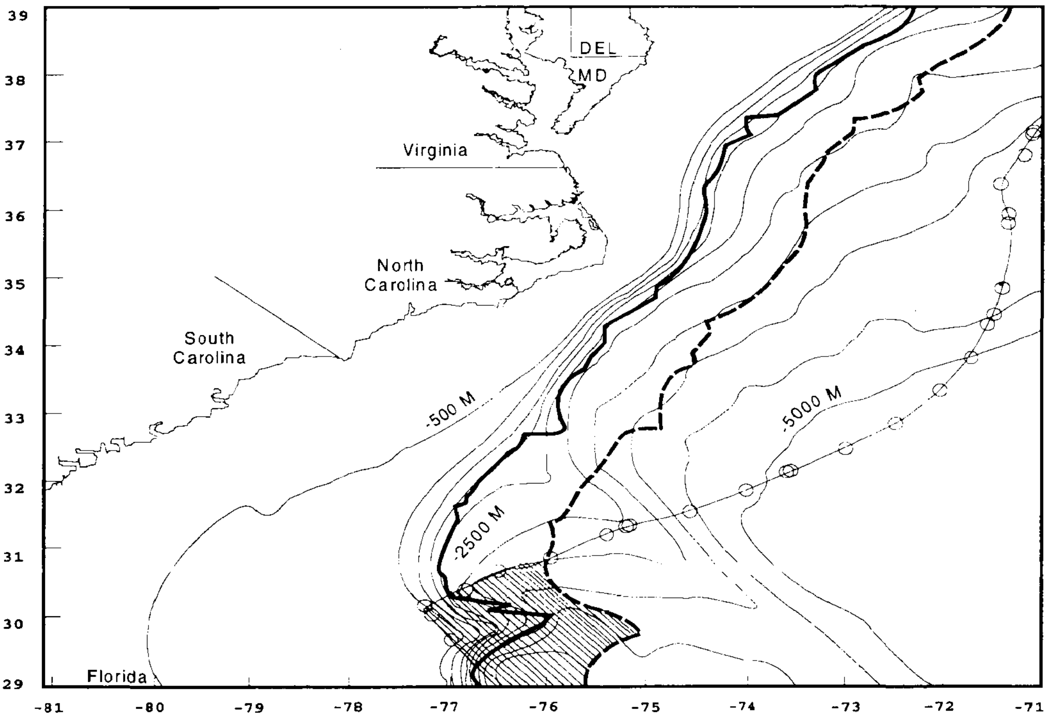

FIG. B2.- Contour map of the NOAA ETOPO5 data with smoothing parameter p=.9990, U.S. Atlantic coast, 10 degrees longitude by 10 degrees latitude (5-nautical mile grid interval; contour interval = 500 metres ; heavy line is a plot of the location of FCS as found by SDG; heavy dashed line is location of the FCS plus 60 nautical miles; line with circles over it is the 200-nautical mile EEZ);

ABBREVIATIONS AND ACRONYMS

CSAPS: A Matlab M-file for doing spline smoothing of digital data in 1-dimension.

D(S): Gradient of the function $S(x,y)$ of 2 variable, i.e., **DS**=(D_xS, D_yS) where D_xS is the partial derivative of S with respect to x and DyS is the partial derivative of S with respect by y.

EEZ: Exclusive Economic Zone, a line 200 nautical miles from coastline. This line determines the seaward extent of a country's mineral rights unless the FCS extends farther seaward than 200 nautical miles. In this case the FCS (plus 60 nautical miles) determines the limits of a country's mineral rights. Mineral Rights can never go beyond 350 nautical miles.

FCS: The Foot of the Continental Slope is "the point of maximum change in the gradient at its base".

H: The Hessian of a function $S(x,y)$ of two variables is a 2 x 2 matrix with entries: $h_{11} = D_{xx}S$, $h_{12}=D_{xy}S$, $h_{21}=D_{yx}S$, $h_{22}=D_{xx}S$.

Matlab: A mathematical software and graphics package by The Math Works, Inc.

M-file: Series of Matlab commands in a script file started by typing its name at a Matlab prompt. M-files can be user defined functions or script files. Matlab has many system and user created M-files to do various mathematical, display, and bookkeeping tasks.

MMS: Minerals Management Service of the Department of the Interior, USA.

NOAA: National Oceanic and Atmospheric Administration.

R.Q.: Rayleigh Quotient a number obtained from a formal mathematical operation on a 2-D vector v and a 2 x 2 Hessian matrix, **H**, as defined in item 4 above where R.Q.(v) = $[v^T\mathbf{H}v]/[v^Tv]$. Superscripting a vector v with "T" means take the transpose of the vector v.

SDG: Surface of Directed Gradient obtained from computing the Rayleigh Quotient of the Hessian of the normalized gradient of $S(x,y)$ in the gradient direction.

SMC: Surface of Maximum Curvature is a mathematical procedure for computing maximum curvature surface for a given input surface $S(x,y)$ of two variables. It is obtained by finding the largest eigen-value of the Hessian for $S(x,y)$.

TCSAPS: A Matlab M-file for doing Spline Smoothing of digital data in two dimensions.