Note

# Spatial Solution To Determine A Trigonometric Point Of High Precision

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## The three "Spheres Of Position" whose radius are the three Geodimetrical distances or "Slope Distance" of GPS

*A, B, C* are three trigonometric points of known coordinates, and $d_1$ $d_2$, $d_3$, are the respective distances measured from the unknown point *O;* the equations of the three spheres, with their centers at *A, B, C* and their radii $d_1$ $d_2$, $d_3$, *are* given by

$$(1)\quad \begin{cases} (X_1-X)^2+(Y_1-Y)^2+(Z_1-Z)^2=d_1^2 & \text{(Sphere with centre at } A\text{)} \\ (X_2-X)^2+(Y_2-Y)^2+(Z_2-Z)^2=d_2^2 & \text{(Sphere with centre at } B\text{)} \\ (X_3-X)^2+(Y_3-Y)^2+(Z_3-Z)^2=d_3^2 & \text{(Sphere with centre at } C\text{)} \end{cases}$$

where the rectangular coordinates of the three known points are

$$(X_1, Y_1, Z_1)$$
$$(X_2, Y_2, Z_2)$$
$$(X_3, Y_3, Z_3)\,.$$

Developing (1) with the statements

$$\begin{cases} X_1^2+Y_1^2+Z_1^2=R_1^2 \\ X_2^2+Y_2^2+Z_2^2=R_2^2 \\ X_3^2+Y_3^2+Z_3^2=R_3^2 \end{cases}$$

they can be written

$$(2)\quad \begin{cases} X^2-2X_1X+Y^2-2Y_1Y+Z^2-2Z_1Z=d_1^2-R_1^2 & \text{(I)} \\ X^2-2X_2X+Y^2-2Y_2Y+Z^2-2Z_2Z=d_2^2-R_2^2 & \text{(II)} \\ X^2-2X_3X+Y^2-2Y_3Y+Z^2-2Z_3Z=d_3^2-R_3^2 & \text{(III)} \end{cases}$$

In Figure 1, we show one of the many configurations of the system of trigonometric points involved in the solution of the system (2); in Figure 2 we have a plain $\pi$ (plain of the sheet) passing for the unknown station and normal to its vertical. The three circumferences with radii $r_1$ $r_2$, $r_3$, are respectively the traces on plain $\pi$ of the three spheres of radii $d_1$ $d_2$, $d_3$, and the points *A', B', C* are the traces of the respective

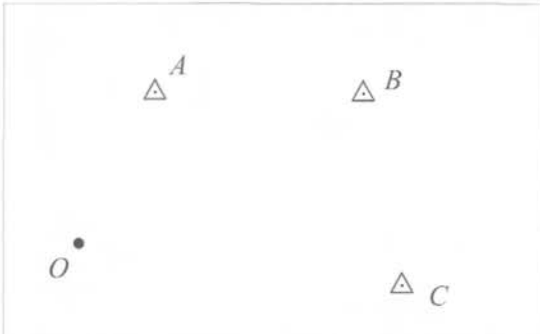


Figure 1

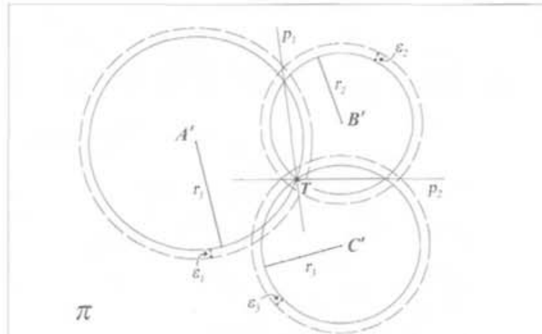


Figure 2

verticals of points *A, B, C* on the same plane.
The continuous line circumferences are relative to the distances without errors, while the dotted ones are relative to the measured distances with a systematic error $\varepsilon$.
The next step is to determine the two radical planes (intersection of the pair of spheres *AB* and the pair *BC)* which on plain $\pi$ determine the traces $p_1$ and $p_2$

$$(3)\quad \begin{cases} 2X(X_1-X_2)+2Y(Y_1-Y_2)+2Z(Z_1-Z_2)=R_1^2-R_2^2+d_2^2-d_1^2 & (\mathrm{II}-\mathrm{I}) \\ 2X(X_3-X_2)+2Y(Y_3-Y_2)+2Z(Z_3-Z_2)=R_3^2-R_2^2+d_2^2-d_3^2 & (\mathrm{II}-\mathrm{III})\ ; \end{cases}$$

the intersection of the radical line (3) (its trace on plain $\pi$ is the point *T)* with one of the three spheres above gives two points, one of them is the sought point.

$$(4)\quad \begin{cases} 2X(X_1-X_2)+2Y(Y_1-Y_2)+2Z(Z_1-Z_2)=R_1^2-R_2^2+d_2^2-d_1^2 \\ 2X(X_3-X_2)+2Y(Y_3-Y_2)+2Z(Z_3-Z_2)=R_3^2-R_2^2+d_2^2-d_3^2 \\ X^2+Y^2+Z^2-2X_1X-2Y_1Y-2Z_1Z=d_1^2-R_1^2 \end{cases} .$$

The equations' system (4) formalizes the intersection of the line (3) with one of the three spheres (see equations (2)) to obtain the station's coordinates; actually it is better to put, in the system (4), the equation of the sphere with the minor radius to have an intersection less sensitive to the errors in the measured distances (radii of the spheres).

We now state

$$\begin{bmatrix} M_1=(X_1-X_2); & N_1=(Y_1-Y_2); & P_1=(Z_1-Z_2); \\ M_2=(X_3-X_2); & N_2=(Y_3-Y_2); & P_2=(Z_3-Z_2); \\ K_1=\dfrac{R_1^2-R_2^2+d_2^2-d_1^2}{2}; & K_2=\dfrac{R_3^2-R_2^2+d_2^2-d_3^2}{2}; & K_3=d_1^2-R_1^2; \end{bmatrix}$$

when replaced in (4) we get:

$$(5)\quad \begin{cases} M_1X+N_1Y+P_1Z=K_1 \\ M_2X+N_2Y+P_2Z=K_2 \\ X^2+Y^2+Z^2-2X_1X-2Y_1Y-2Z_1Z=K_3 \end{cases} .$$

Multiplying the first equation of (5) by $N_2$ and the second by $N_1$ and solving the system with respect to *X:*

$$\begin{cases} N_2M_1X+N_2N_1Y+N_2P_1Z=N_2K_1 \\ N_1M_2X+N_1N_2Y+N_1P_2Z=N_1K_2 \end{cases}$$

$$X = \frac{N_2K_1 - N_1K_2}{N_2M_1 - N_1M_2} - \frac{N_2P_1 - N_1P_2}{N_2M_1 - N_1M_2} Z \quad ;$$

stating

(6a)
$$D = N_2M_1 - N_1M_2$$
$$\alpha_1 = \frac{N_2K_1 - N_1K_2}{D}$$
$$\beta_1 = \frac{N_2P_1 - N_1P_2}{D} \quad ,$$

it can be written:

(6b) $$X = \alpha_1 - \beta_1 Z \; .$$

Likewise, multiplying now the first equation of (5) by $M_2$ and the second by $M_1$ and solving the system with respect to *Y:*

$$\begin{cases} M_2M_1X + M_2N_1Y + M_2P_1Z = M_2K_1 \\ M_1M_2X + M_1N_2Y + M_1P_2Z = M_1K_2 \end{cases}$$

$$Y = \frac{M_1K_2 - M_2K_1}{M_1N_2 - M_2N_1} - \frac{M_1P_2 - M_2P_1}{M_1N_2 - M_2N_1} Z \quad ;$$

stating

(7a)
$$D = M_1N_2 - M_2N_1$$
$$\alpha_2 = \frac{M_1K_2 - M_2K_1}{D}$$
$$\beta_2 = \frac{M_1P_2 - M_2P_1}{D} \quad ,$$

it follows

(7b) $$Y = \alpha_2 - \beta_2 Z \; .$$

At this point, replacing (6b) and (7b) in the third equation of the system (5), we get an equation which is function of Z

(8) $$(\alpha_1 - \beta_1 Z)^2 + (\alpha_2 - \beta_2 Z)^2 + Z^2 - 2X_1(\alpha_1 - \beta_1 Z) - 2Y_1(\alpha_2 - \beta_2 Z) - 2Z_1 Z = K_3 \; ;$$

developing the (8), we get:

$$\alpha_1^2 + \beta_1^2 Z^2 - 2\alpha_1\beta_1 Z + \alpha_2^2 + \beta_2^2 Z^2 - 2\alpha_2\beta_2 Z + Z^2 - 2X_1\alpha_1 + 2X_1\beta_1 Z - 2Y_1\alpha_2 + 2Y_1\beta_2 Z - 2Z_1 Z = K_3$$

and again

$$\left(\beta_1^2 + \beta_2^2 + 1\right)Z^2 - 2\left(\alpha_1\beta_1 + \alpha_2\beta_2 - X_1\beta_1 - Y_1\beta_2 + Z_1\right)Z + \alpha_1^2 + \alpha_2^2 - 2\left(X_1\alpha_1 + Y_1\alpha_2\right) - K_3 = 0.$$

With the statements

(9)
$$a = \left(\beta_1^2 + \beta_2^2 + 1\right)$$
$$b = \left(\alpha_1\beta_1 + \alpha_2\beta_2 - X_1\beta_1 - Y_1\beta_2 + Z_1\right)$$
$$c = \alpha_1^2 + \alpha_2^2 - \left(2X_1\alpha_1 + 2Y_1\alpha_2 + K_3\right)$$

we can write the quadratic equation

(10) $aZ^2 - 2bZ + c = 0$ .

The two solutions of equation (10) are

$$Z_{O1} = \frac{b + \sqrt{b^2 - ac}}{a}$$

$$Z_{O2} = \frac{b - \sqrt{b^2 - ac}}{a}$$

that, placed in (6b) and in (7b), will give the rectangular coordinates of two stations of which one is the sought point.

$X_{01}, Y_{01}, Z_{01}$ *and* $X_{02}, Y_{02}, Z_{02}$

To demonstrate the validity of the method, let us proceed with a rigorous, hence with no errors, numerical example, with the known station used only to calculate the three distances exactly.

Let the coordinates ED50 of the three points observed be

$$A\begin{cases}\varphi = 40°19'28.197''\\ \lambda = 15°42'25{,}980''\\ h = 1550{,}10\ m\end{cases} \qquad B\begin{cases}\varphi = 40°37'53{,}590''\\ \lambda = 15°24'46.535''\\ h = 902{,}43\ m\end{cases} \qquad C\begin{cases}\varphi = 40°01'23{,}331''\\ \lambda = 15°21'01{,}374''\\ h = 553{,}25\ m\end{cases}$$

and the station's coordinates

$$O\begin{cases}\varphi = 40°22'02{,}167''\\ \lambda = 15°01'40.875''\\ h = 370{,}43\ m\end{cases} \quad ;$$

transforming all of them into rectangular coordinates, we get

$$A\begin{cases}X_1 = 4688981{,}44521\\ Y_1 = 1318650{,}52709\\ Z_1 = 4106593{,}80372\end{cases} \qquad B\begin{cases}X_2 = 4673875{,}09104\\ Y_2 = 1288534{,}22517\\ Z_2 = 4132114{,}68460\end{cases} \qquad C\begin{cases}X_3 = 4717188{,}64338\\ Y_3 = 1294936{,}23597\\ Z_3 = 4080378{,}19263\end{cases}$$

$$O\begin{cases}X = 4700444{,}85008\\ Y = 1261944{,}54953\\ Z = 4109450{,}31879\end{cases} \quad .$$

With the foregoing symbolism, we calculate the exact distances and we get

$OA = d_1 = 57923{,}54634\ m$

$OB = d_2 = 43893{,}46675\ m$

$OC = d_3 = 47053{,}10306\ m$

$R_1 = 6370988{,}84592$

$R_2 = 6370227{,}67119$

$R_3 = 6370103{,}19754$ ;

using the previous values in the equations' system (4) we get:

$$\begin{cases} M_1 = 15106{,}35417; & N_1 = 30116{,}30192; & P_1 = -25520{,}88088; \\ M_2 = -28207{,}19817; & N_2 = 23714{,}29112; & P_2 = 26215{,}61108; \\ K_1 = 4134895637{,}85680; & K_2 = 5071492379{,}42728; & K_3 = -40577873946346{,}30000; \end{cases}$$

following the declared (6a) and (7a) we get

$$D = 1207732976{,}92717$$
$$\alpha_1 = -45273{,}6472474292$$
$$\beta_1 = -1{,}1548304832$$
$$\alpha_2 = 160006{,}876035811$$
$$\beta_2 = -0{,}2681472187 \quad ;$$

then, replacing these last ones in (9) we get:

$a$ = 2,4055363758
$b$ = 9884542,9722487
c = 40616383922090

and consequently we have the two solutions

$Z_{01}$ =4109450,31880
$Z_{02}$ =4108710,97906

which, replaced in (6b) and (7b), give the rectangular coordinates of the two points, of which one is just the station

| | |
|---|---|
| $X_{01}$ = 4700444,85009 | $X_{02}$ =4699591,03802 |
| $Y_{01}$ =1261944,54954 | $Y_{02}$ =1261746,29764 |
| $Z_{01}$ =4109450,31880 | $Z_{02}$ = 4108710,97906 . |

To transform rectangular coordinates into geodetic coordinates, we use one of our formulary of rapid convergence

$$r = \sqrt{X^2 + Y^2}$$
$$\varphi_o = \tan^{-1}\left(\frac{Z}{r\left(1-e^2\right)}\right)$$
$$N_o = \frac{a}{\sqrt{1-e^2 \operatorname{sen}^2 \varphi_o}}$$
$$\phi = \tan^{-1}\left(\frac{Z + e^2 N_o \operatorname{sen}\varphi_o}{r}\right)$$
$$\begin{cases} \varphi = \tan^{-1}\left(\dfrac{\operatorname{sen}\phi - e^2 \operatorname{sen}\varphi_o}{\cos\phi\left(1-e^2\right)}\right) \\ \lambda = \tan^{-1}\left(\dfrac{Y}{X}\right) \\ h = \dfrac{r}{\cos\varphi} - N \quad \text{(where } N = f(\varphi)\text{)} \end{cases}$$

and we get the coordinates of the two stations, one of which is the real one.

$$O_1\begin{cases}\varphi = 40°22'02,167''\\ \lambda = 15°01'40.875''\\ h = 370,43\ m\end{cases} \qquad O_2\begin{cases}\varphi = 40°22'02,230''\\ \lambda = 15°01'42.143''\\ h = -775,87\ m\end{cases}.$$

In the case at hand, the <u>negative elevation</u> excludes the point $O_2$; alternatively the two elevations could be both positive, but a rough estimate of the elevation of the zone (contour line, for instance) will be sufficient to choose the right point.

Repeating the procedure by introducing a systematic error of 1cm in the measured distances, we get

$OA = d_1 = 57923,55634\ m$
$OB = d_2 = 43893,47675\ m$
$OC = d_3 = 47053,11306\ m$

$R_1 = 6370988,84592$
$R_2 = 6370227,67119$
$R_3 = 6370103,19754$

and using the new values in the equations' system (4), we get

$$\begin{bmatrix} M_1 = 15106,35417; & N_1 = 30116,30192; & P_1 = -25520,88088;\\ M_2 = -28207,19817; & N_2 = 23714,29112; & P_2 = 26215,61108;\\ K_1 = 4134895497,55601; & K_2 = 5071492270,72285; & K_3 = -40577873945468,40000; \end{bmatrix}$$

then, following the statements (6a) and (7a), we get

$$D = 1207732976,92717$$
$$\alpha_1 = -45273,6472916098$$
$$\beta_1 = -1,1548304832$$
$$\alpha_2 = 160006,8713993390$$
$$\beta_2 = -0,2681472187 \quad .$$

The (9), when these last values are replaced in, take to

$a = 2,4055363758$
$b = 9884542,97354298$
$c = 40616383932099,9$

so that the coordinates of the two points are:

$X_{01} = 4700445,26129$ and $X_{02} = 4699590,62798$
$Y_{01} = 1261944,64039$ and $Y_{02} = 1261746,19780$
$Z_{01} = 4109450,67491$ and $Z_{02} = 4108710,62403$ ;

the same, in geodetic coordinates, are:

$$O_1\begin{cases}\varphi = 40°22'02,167''\\ \lambda = 15°01'40.874''\\ h = 370,\mathbf{98}\ m\end{cases} \quad \text{and} \quad O_2\begin{cases}\varphi = 40°22'02,299''\\ \lambda = 15°01'42.143''\\ h = -77\mathbf{6,42}\ m\end{cases}. \qquad (^1)$$

([1]) - The bold numbers of the values of the last coordinates enhance the consequence of the introduced systematic error.

Comparing the coordinates of the two points, the height of the second reveals that the point $O_1$ is the good one. As we can see, the geodetic accuracy in planimetry is kept, while the elevation is quite different from the real elevation of the station. Anyway, that difference is acceptable in a trigonometric levelling.

In conclusion, with a systematic error on the distance of about 1 *cm,* we can state that it does not affect the geodetic accuracy of the point; **besides, the closer the distance values become, the more the influence of the error on the sought point lessens.**

## Bibliography


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