Article

Re-surveying Decision Support Methodology

Fuzzy Logic as an instrument for establishing area selection and priority

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Abstract

Decision making for hydrographic re-surveying is not a simple task. In order to make such decisions, authorities must find a way to combine many types of information to select the areas that should be re-surveyed. In this article the authors will analyze how to organize the necessary information and describe the main types of indicators for the re-surveying decision. After that, the authors will show the advantages of using Fuzzy Logic to combine them into criteria. Finally a suggestion of a re-surveying decision support methodology will be given, followed by a case study and some brief conclusions.

Résumé

Les prises de décision relative à l'exécution de nouveaux levés hydrographiques ne constituent pas une tâche simple. Afin de prendre de telles décisions, les autorités doivent trouver un moyen de combiner de nombreux types d'informations pour choisir les zones qui doivent être à nouveau hydrographiées. Dans cet article les auteurs analysent comment organiser les informations nécessaires et décrivent les principaux types d'indicateurs quant à la décision d'effectuer de nouveaux levés. Après cela, les auteurs s'attachent à montrer les avantages qu'il y a à utiliser Fuzzy Logic pour les combiner en critères. Enfin, une suggestion concernant une méthode de soutien aux prises de décisions liées à l'exécution de nouveaux levés sera faite, suivie d'une étude de cas et de quelques brèves conclusions.

Resumen

La toma de decisiones para un nuevo levantamiento hidrográfico no es una tarea sencilla. Para tomar tales decisiones, las autoridades tienen que encontrar la manera de combinar muchos tipos de información, para seleccionar las áreas que deberán ser levantadas nuevamente. En este artículo los autores analizarán cómo organizar la información necesaria y describirán los principales tipos de indicadores para tomar la decisión de un nuevo levantamiento. Después de esto, los autores mostrarán las ventajas de utilizar una "Lógica Confusa" para combinarlos en los criterios. Finalmente, se sugerirá una metodología de apoyo para la decisión de un nuevo levantamiento, seguida del estudio de un caso y de algunas breves conclusiones.

Introduction

In the last edition of IHO Special Publication S-55 – Status of Hydrographic Surveying and Nautical Charting Worldwide – the International Hydrographic Organization notes that one of the major skill deficiencies currently faced by Coastal States is the difficulty in planning a prioritised survey program. That also involves routines of re-surveying, particularly to assure safe access to ports and navigationally complex areas (IHO 2004).

Establishing a priority for such re-surveys, especially in countries with an extensive coastal area and a large number of ports, is essential, due to the cost, time, navigation safety and the political-economical elements involved. However, this is not an easy task for the relevant authorities. It is fundamental to establish appropriate criteria to determine where re-surveying is necessary, and then to support the subsequent decision about where it should be carried out first. This would normally require a detailed knowledge of the real situation in those areas. This includes not only the information about the previous hydrographic surveys performed but also the technical survey requirements for each area according to the relevant standards such as IHO S-44 Standards for Hydrographic Surveys, and data describing such things as rates of change of the seabed and the volume of maritime traffic using or expected to use the area in the future.

Analysts could achieve this by selecting all the available information about the area and choosing the most relevant factors to indicate the necessity of re-surveying. Subsequently, they could make a thematic classification of these indicators, establishing sets for each one in accordance with the need. These sets could be combined by mathematics and rules of logic in order to achieve criteria for re-surveying.

However there are some problems with such a methodology. Most of the data that could be used as indicators, such as the age of the survey, are expressed in continuous scales without a clear border of change from one set to another, and cannot be truly expressed in a realistic way by a conventional Boolean approach. Furthermore the number of rules of logic necessary to combine a large number of indicators and sets to reach reasonable criteria for a re-surveying analysis would be too great for relatively simple examination and decision making. Therefore a better alternative other than traditional thematic classification must be used.

Fuzzy Inference Systems (FIS), based on Fuzzy Logic and Sets Theory, have been applied successfully to deal with these kinds of problems in many control and decision support systems. Their success is mainly a result of their ability to mimic aspects of human perception and thinking and also because their relative simplicity of use.

However before we start to explain how a FIS works and how to use it as a re-surveying analysis tool, it is important to identify what information we will use and how to extract the main indicators for establishing the necessity and priority for re-surveying.

Data Preparation

It is important to determine what information will be significant indicators in any analysis. If we do not bear in mind what kind of information will be used in our criteria, gathering such data could be a waste of time. We must also consider the difficulty in setting the geographic border limits of these data if we intend to use them in the re-surveying indicators.
When we look into the status of hydrographic surveys worldwide as presented by the IHO (IHO 2004), a principal question that arises for Hydrographic Authorities is how they can determine which areas are or are not adequately surveyed. The simple answer would seem to be to compare existing surveys with a recognized standard. However, this is not possible when we do not have the necessary information about the pre-existing surveys or if we do not have a standard to provide a comparison.

Furthermore, the creation of a comprehensive hydrographic database containing all the existing, mainly bathymetric, information for existing surveys in a given area is still far away from a reality for most of the World's Hydrographic Services. It would demand not only a significant data storage capacity, but also large personal and time consuming efforts to digitise all the older surveys carried out before digital processing and storage. On the other hand, the creation of a database using the surveys' metadata is something that is easier to achieve in a short period of time, which makes it attractive to use the metadata as indicators instead.

First of all, it is important to determine which surveys, according to standards and age limits, can be used to determine the priority for re-surveying. This will save a lot of unnecessary work when inserting the information in the database. When digitising existing metadata, from a Report of Survey, it may be beneficial to follow the data transfer standards established by the IHO in S-57 - Transfer Standard for Digital Hydrographic Data (IHO 2000) to ensure that the resultant metadata is easily available for exchange and for subsequent use both for this analysis as well as for use by other users.

Establishing what should be the ideal, or minimum technical standards applied to each part of an area is the second preliminary input. In this regard, Hydrographic Services must determine which are the geographic areas where the different S-44 specifications should apply. Knowledge of the limits of the harbours and the "minimum under-keel clearance" are essential pieces of information in specifying these areas. As well as the technical criteria specified in IHO S-44, analysts should study the necessity to create areas where specific and more rigid criteria are used, due to their military, strategic, political or economic relevance.

Areas that define other important indicators will also need to be identified. Areas defining such things as the rate of seabed change, volume of maritime traffic, and classification of an area according to parameters such as the number of nautical accidents, or navigation safety relevance may be important.

With the information about the existing reality of the maritime area, it will be possible to determine the most important indicators that should be used for the re-surveying decision problem.

Re-surveying Indicators

The first thing to do if we want to identify an area for re-surveying is to establish how up to date or current the information is. In these kinds of problems, the passage of time is an essential element to consider. It is the main dimension used to vary some of the indicators. The age of the survey is, then, the first and most clearly identifiable indicator.

When we classify a survey by its age, we can take into account the potential for changes in an area in both physical, economic and other terms. This indicator can also reflect the technical evolution of the equipment used to survey the area. In this case we use time as the generalized factor that can account for those parameters that we cannot use directly as indicators in their own right. However, we must stay alert not to either overestimate, or minimise its relevance in establishing the classification and weight of the indicators present in the criteria.

The other types of indicators immediately linked with the update concept are the spatial modification indicators. The main reasons to re-survey an area are to account for any changes that have happened to it. These changes may have happened due to human or natural action, both over and under the sea. The establishment of a rate that measures these changes is a difficult task. Although changes in the landward area of a chart are important, the most significant changes that drive the need for any re-surveying or alteration in the seabed topography. There are several ways to measure it from comparing the bathymetric data, measured in different surveys, to methodologies involving sediment profiling. A large effort must be made to establish and update such data periodically if we truly want to discover the necessity to re-survey a given area.

As we have explained before, it is also important to compare the existing survey information with standards so we can determine if the area was adequately surveyed in the first place. To establish this we can use Technical Indicators. These indicators can be obtained by comparing the existing technical information in the surveys' metadata with the standards established by the relevant Hydrographic Service. The obvious standards to use are those expressed in IHO S-44: positioning precision, depth precision, line spacing and bottom coverage.

There are several ways to combine the metadata with the standards. The easiest way is to determine a ratio between the metadata and the specified standard. Fixing this information for line spacing and bottom coverage will probably not be a problem. But sometimes finding this number for the other ratios (depth and positioning measurement) is not an easy task, particularly if we consider old surveys, where some of these metadata, for instance the tide reference station information, and the method of determining sound velocity, are not available. In these cases, an exact number will not be achieved, making it more difficult to use such information in a criterion. Although we may use some assumed values based on technical references, doing this in the age indicator must be considered carefully. If we choose to proceed in this way, the technical evolution of the measurement equipment and the period when we start to use them should be used to establish the age indicator classification of the survey.

Finally, after determining if the hydrographic/bathymetric information of the areas is updated, or not, it is also important to classify such areas, according to criteria, identifying those that should be re-surveyed first. To help us in this job we can use the priorisation indicators. Several relevance indicators can be used to construct such criteria. Probably the most visible, and important, are those that are obtained using economic and navigation safety information. To establish such indicators data such as port cargo and vessel traffic volumes can be used. The competent authorities and specialists must choose them considering the available data and the difficulty of combining their degree of importance into a rate, or even a class.

With the main indicators specified, we can now discuss the basic concepts of Fuzzy Logic.

Fuzzy Sets

The concept of "middle-age" is vague. People are not young when they are 34 years old and suddenly in their 35th birthday they become middle-aged. What happens in real life is a progressive change from a point when we are sure that people are not middle-aged to the point when we are sure that they are. This also applies to other vague concepts such as near, far, big, small, and also for quantifiers such as many and few. Many of these continuous scale descriptors are used in classifying such things as temporal, geo-biophysics and socio-economic data. This, of course, can include hydrographic survey data, especially in terms of its currency and utility.
Fuzzy sets Theory was conceived in the 1960´s by Lofti Zadeh (Zadeh 1965). It is the base of the fuzzy logic and its main objective is to generalise the idea represented by the conventional or ordinary sets theory, approaching the imprecision and vagueness of human reasoning (Kosko B. 1992).

Unlike the conventional sets, where an element belongs or not to a set, in the fuzzy sets a given element is associated with a set by a degree of membership (μ) that varies from zero to one. This type of treatment allows that the transition between the conditions of belonging or not belonging to a set do not occur in a crispy, abrupt way, but progressively, as shown in Figure 1.
For a more detailed mathematical description of Fuzzy Set Operations see Appendix 1 of this paper.

Fuzzy Logic

These set theoretic operations provide the fundamental tools for the logic propositions and rules. In Boolean logic, the intersection can be viewed as the logic operation "and", the union as "or", and the complement as "not". The same logic propositions and inferences applicable for the crispy sets can be used for the fuzzy sets. This kind of logic is then named "fuzzy logic".

In many cases, fuzzy sets can classify the same phenomenon more adequately and in fewer numbers of sets than conventional logic, as seen in Figure 2. Therefore, its usage becomes very advantageous

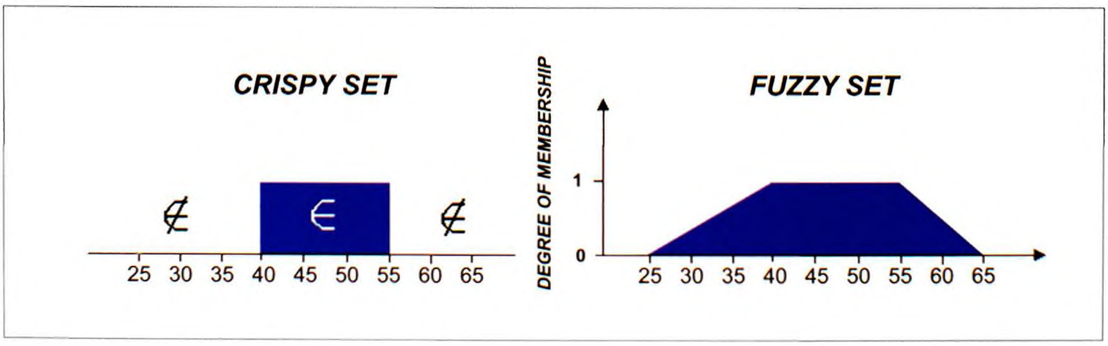

Figure 1: Middle-aged Persons Set.

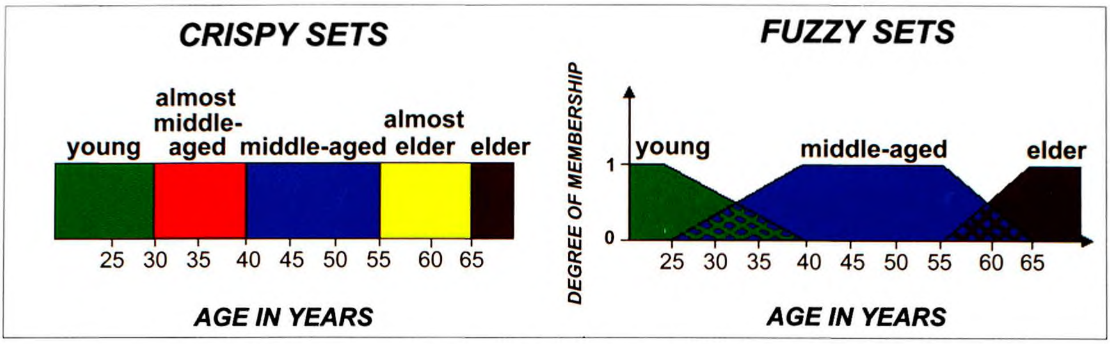

Figure 2: Age Classification Sets.

when we have to combine a large number of antecedents into a predicate because it avoids the creation of many unnecessary sets that would raise a great number of rules and, as a consequence, the complexity of the logic inference system.

For example, if we want to combine the age classification sets in Figure 2 two by two, the number of rules would be much greater if we use crispy sets (25 rules) instead of fuzzy sets (9 rules). This difference becomes greater as we raise the number of sets or grow the number of antecedents to combine. For instance, if we want to combine the same age classification five by five, the difference of the number of rules would grow to $5^5 - 3^5 = 2882$.

Fuzzy logic also allows us to deal with uncertain situations where we are not sure if we must use an "and" or a "or", as in the affirmation: "The person who smokes or/and has obesity problems has a big risk of a heart stroke". For these situations we use an operator that satisfies both the t-norms (AND) and t-conorms (OR). It is named "Gamma Operator", and is defined from the algebraic product and sum concepts as:

$$\text{Gamma} = (\text{algebraic sum})^{\gamma} * (\text{algebraic product})^{1-\gamma} \quad (1)$$

Where the values of γ varies from 0 to 1. For $\gamma=0$ the result is equal to the fuzzy algebraic product, and for $\gamma=1$ the result is the same as the fuzzy algebraic sum (Bonham-Carter 1999).

Fuzzy Inference System - FIS

Fuzzy Inference System (FIS), also named Fuzzy Logic Controller, is a technique that uses fuzzy logic for Decision and Support Systems and its use is increasing because it is an effective and accurate way to describe human perceptions of decision-making problems (Turban 2005).

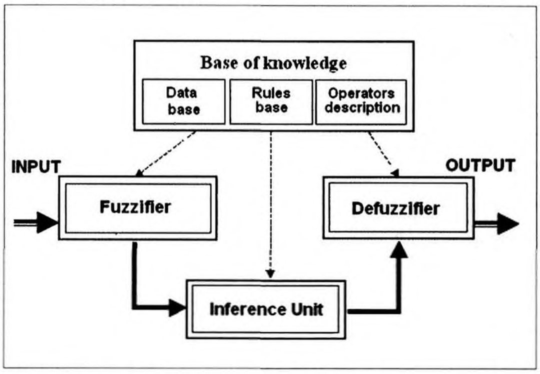

Figure 3: Fuzzy Inference System.

As we can see in Figure 3, the FIS is based on the simple input, process and output flow concept. It consists basically of inputs that are associated with a fuzzy set by a specific degree of membership in a process called fuzzification. The sets are then combined in the inference unit through logic rules in order to generate results associated with output sets.

We must then combine the resulting output fuzzy sets from all the rules in order to have one single fuzzy set. This step is made by a process called aggregation. The most used aggregation methods in the computer programs are the "maximum" and the "algebraic sum" operations (Maranhão 2005).

Because normally we can not express an element in the physical world by using a fuzzy set unassociated with a degree of membership, we must somehow generate an exact value that represent or summarise in the best possible way the information that is present in the fuzzy set originated by the aggregation process (Oliveira Jr. 1999). Therefore, this fuzzy set is transformed in a single crispy value, unassociated with fuzzy sets, in a process called defuzzification.

The defuzzification process can be made by several mathematical methods. The most popular are:

- Mean of Maximum Method (MOM): that uses the abscise of the middle point between the values that have the greater degree of membership inferred by the rules; and
- Centre of Gravity Method (COG): gives the mass centre abscise associated with the aggregation resulting set graphic. The output is the value that divides the area, under the pertinence function, into two equal parts.

This last method, also named as Centre of Area (COA) or Centroid Method, is largely used and has very satisfactory results. But when we use discrete universes, its implementation becomes difficult because of the necessary numeric integration, demanding, normally, a substantial computational effort.

There are several types of FIS models that are different in premise terms and control action representation and in the operators for the controller implementation used in the "inference unit". The choice of what will be the best FIS model to be used depends on what type and precision of information will be used in the system.

The SIF types can be divided into classic models, as the Mandani and Larsen models, and in interpolation models, which use the Gamma Operator, as the Takagi-Sugeno and Tsukamoto models. We suggest for the hydrographic survey selection/priority problem, initially, the usage of the Mandani model for its simplicity and large implementation in the existent software.

Methodology for Re-surveying Decision Using FIS

The methodology suggested for establishing the selection and priority of areas to hydrographic re-surveying is shown in Figure 4.

Initially all the areas of the previous survey areas and their metadata must be clearly defined. It is also fundamental to delineate the area limits of the other information that will generate indicators, such as the ideal or minimum technical standards, and the seabed variation rate.
Next, we must intersect the survey areas with the other indicator areas. This intersection should be done in a Geographic Information System (GIS) to guarantee a better visualization and a repeatable result.

At this point we must pay attention to a very important problem. Many of the surveys overlap or are coincident in the same area. It is therefore very important to make the intersection of the survey areas with the other indicator areas individually, survey by survey. By using this procedure we will avoid mixing the metadata of different survey records and the consequent errors. As a result of this combination methodology we can guarantee that each new area, obtained by the intersection, corresponds to a specific record in a database with all the individual information necessary to establish its update necessity and priority grade.

After this, it is necessary to use a FIS to select the areas that need survey update. This can be done by combining together the age indicator with the special modification and technical indicators. The number of

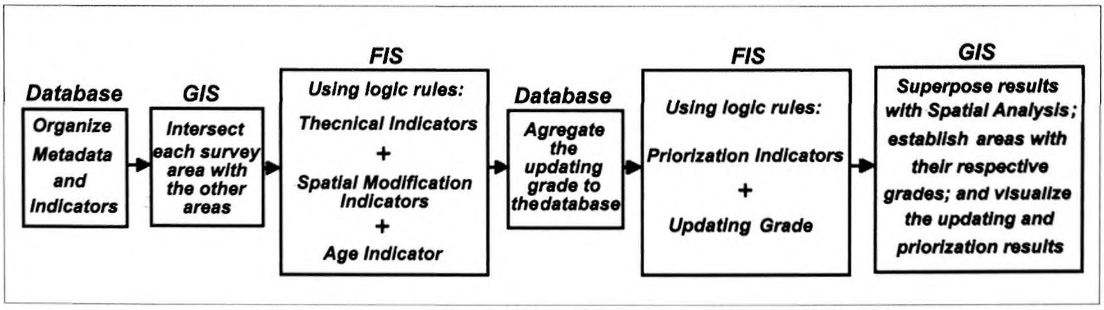

Figure 4: Methodology for re-survey decision.

indicators and sets in each one should not exceed four, otherwise the number of rules will be excessive, making it difficult to establish and modify them if necessary. If the number of indicators is higher, we suggest first that the correlated ones should be grouped in a single indicator for technical specifications and/or spatial modification. This grouping can be done in a preliminary FIS.

Accompanying the reality of the main metadata and information available, we suggest the seabed variation rate should be used for the criteria of spatial modification indicator; as technical indicators, the ratio of line spacing and bottom coverage obtained by comparing the metadata of the survey and the established ideal standards; and the age indicator, where the establishment of fuzzy sets must also consider other spatial modification and technical parameters, such as evolution in precision and measurement techniques of position and bathymetric data through the years.

All the sets and rules used in the FIS must be established carefully, taking advice from the relevant subject matter experts. This is because the construction of the sets will affect the rules and vice-versa. For example, a country that started to use side-scan sonar equipment in 1981 should not have the same sets and rules as another that started to use the equipment in 1985. The sets and rules must also be revised periodically in order to mirror any changes both in priority and in the reality of the area.

It is also important to establish the other parameters of the FIS. We have to select the function that will represent the union and the intersection, that will be used as "or" and "and" in the fuzzy inferences. We must also choose what will be the aggregation process and the defuzzyfication process. Fixing such parameters is not an easy task. We strongly recommend that those involved in this job study carefully the characteristics, advantages and disadvantages of the methods before defining the characteristics of each FIS that will be used in the re-surveying decision support methodology.

Once a score in the Updating FIS has been obtained, which expresses the updating need for each record, this information can be used together with the prioritisation indicators in a new FIS to establish the prioritization of the area. It is also important that the sets and indicator numbers are not too big. We suggest using as prioritization indicators the economic and navigation safety relevance rates, which must be specified and established according to the available information and reality of each area in study. At the end of this process, it is necessary to use the GIS again in order to take away the superposition of information, and really show the need of updating and prioritization in the areas. To make it possible, we must establish a priority for plotting the result areas, overlaying the ones with greater updating and prioritisation grades by those with the smaller ones. For example, if we have two surveys that have an intersection in a specific area and one has no need of updating while the other has it, their intersection will accompany the more updated one and will not need updating. It is important, therefore, to determine the coordinates of these new areas by spatial analysis and to establish what their grade of updating and prioritization will be.

Case Study

We have chosen Guanabara Bay (in Brazil) and its surrounding area in order to provide a small example of the usage and possibilities of the methodology. We used CARIS GIS 4.4, MATLAB 7.0 (for constructing and running the FIS) and Microsoft Access database as software in this case study.

In the Guanabara Bay area, there were about 250 hydrographic surveys from 1960 to 2005, which were, are or can be used in the compilation of nautical charts. Although there are different surveys covering exactly the same area, it was usually impossible to select, due to their different characteristics, which would have the best updating score. Furthermore sometimes there are different equipments, line spacing and bottom coverage for different areas in the same survey, growing the number of registers that have to be inserted into the database. Therefore the number of previous surveys registers inserted at the database reached 357, instead of the original 250 surveys.

After that, we had to specify the ideal standards for the areas and the spatial modification indicator. For the standards, we considered strategic, economic and hydrographic studies to define the area borders and the specifications. For the spatial modification indicator, we used the most recent sedimentation studies as the basis to define the areas.

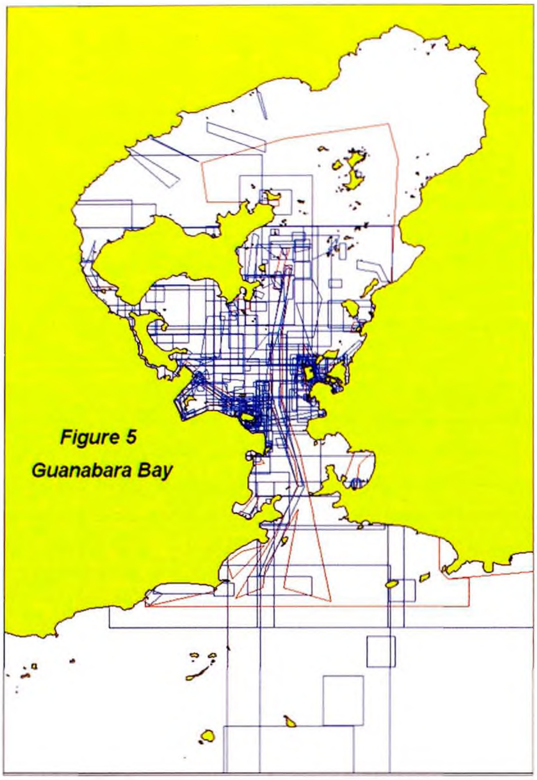

Figure 5: Guanabara Bay: surveys, sedimentation rates and ideal standards areas borders.

When we finally intersected the survey areas individually at the GIS with the ideal standards and sedimentation areas, 889 registers were made. The borders of these areas can be seen in Figure 5.

The same procedure was used to choose the re-surveying prioritisation indicators, totalizing a final number of 1673 registers of areas in the database. The specialists had to fix, then, the set, rules and other FIS parameters. They decided to use survey's age (4 sets), line spacing ratio (3 sets), bottom coverage comparison with ideal standards (4 sets) and sedimentation rate (3 sets) as indicators for the updating of the areas. Excluding impossible situations, the Updating FIS had 72 rules with 4 set classes of output. For instance, if we were using a normal thematic classification instead of fuzzy sets we would have 514 rules for the same situation.

The resulting updating grades (4 sets) were then combined with a strategic relevance classification (3 sets), the economic movement of the ports and terminals and their influence areas (3 sets) and a navigation relevance classification (4 sets) by 57 rules to obtain the prioritisation grade, divided in 4 set classes.

For both FIS specialists, we have decided to use as parameters for intersection the minimum, for aggregation the maximum and for defuzzification the centroid methods.
The results of the methodology are expressed numerically at Table 1 and can be visualized in Figures 6 and 7.

Assuming that the parameters of the Updating FIS are correct, the high percentage of areas that need re-surveying (areas classified as "not updated" and "little updated") may have been caused by two main reasons. The first is the short period of time to re-survey all maritime areas, mainly port and shallow waters, adequately after the modification of the survey standards implemented by the last S-44 edition in 1998. The second is an inadequate frequency or possible degradation in the frequency in which the area is surveyed, demanding, therefore, more investments from the competent authorities in surveying campaigns at the area. To identify which is the main cause a continuous study of the evolution in the updating class area's percentage should be made.

The second FIS achieved success in establishing a priority for re-surveying as we can see by the increasing percentages of the prioritization classes. However, most of the "very high priority areas" are chan-

Total Maritime Area (m²)			**681937510**		
Updating			**Priority**		
Class	**Area (m²)**	%	**Class**	**Area (m²)**	%
Not Updated	155031674	22.734%	Very High Priority	15193568	2.42%
Little Updated	344310249	50.490%	High Priority	81962070	13.06%
Reasonably Updated	154076961	22.594%	Medium Priority	187969255	29.96%
Almost Updated	28518627	4.182%	Low Priority	342257617	54.55%

Table 1 - Guanabara's Bay Re-surveying Support Information Results

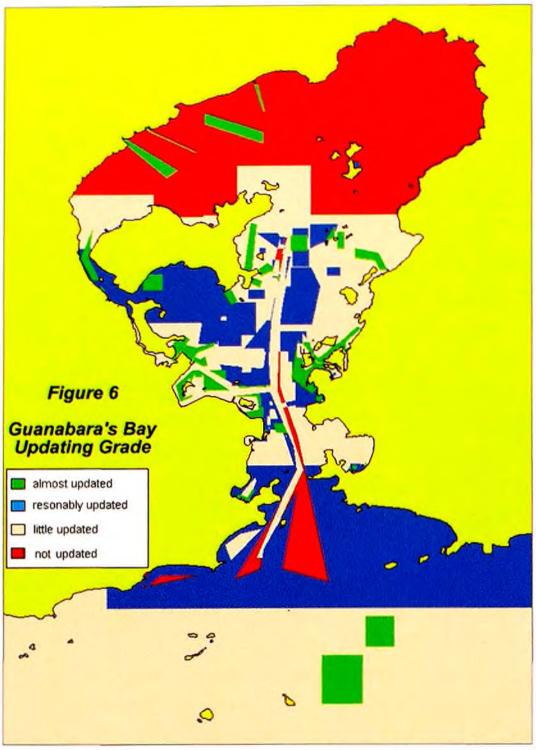

Figure 6: Survey updating grades of Guanabara's Bay.

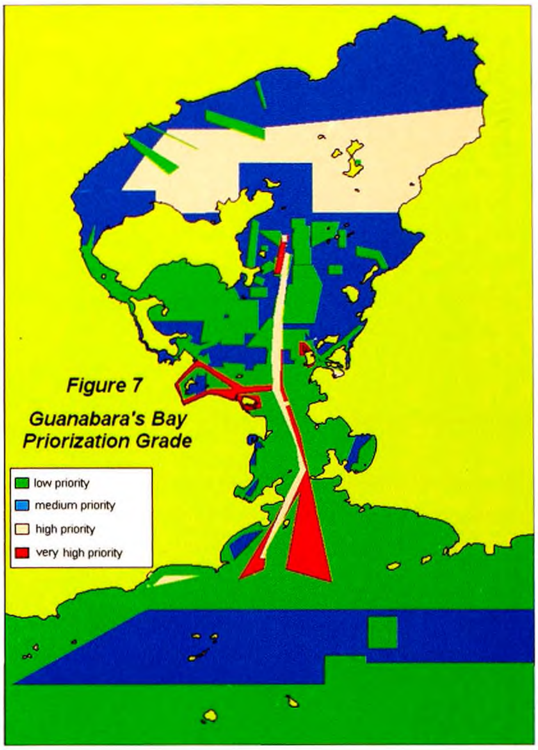

Figure 7: Re-surveying priorities.

nels and port areas that are regularly re-surveyed. Therefore, if we want to identify which of these areas should be more frequently surveyed, we should raise the number of sets or even make this study analyzing only the resulting number of the prioritisation grade instead of dividing it into classes.

Conclusions

The suggested decision support methodology for re-surveying using a FIS together with a GIS achieves a viable and easy way to treat spatial information and establish the necessary criteria to select and prioritize areas for hydrographic re-surveying.

In order to do that, the knowledge of all metadata of the previous surveys in the area is fundamental.
It is also important to do preliminary studies, collect and "spatialise" other necessary data to establish the indicators.

The indicators, sets and rules should be established considering each specific reality, then revised and changed as the situation dictates. Analysts must pay particular attention to the other FIS characteristics and analyse what kinds of parameters they will choose.

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Appendix 1

Fuzzy Set Operations

As conventional sets, there are specifically defined operations for combining and modifying fuzzy sets. Following the conventional fuzzy set operations, initially proposed by Zadeh, the basic operations of two Sets "A" and "B" respectively with elements "x" and "y" are:

Intersection – $A \cap B = \min(\mu A[x], \mu B[y])$ (2)
Union – $A \cup B = \max(\mu A[x], \mu B[y])$ (3)
Complement – $\sim A = 1 - \mu A[x]$ (4)

Since fuzzy sets are not crisply partitioned, in the same sense as Boolean sets, these operations are applied at the truth membership level. As a consequence of a fuzzy set's somewhat fluid characteristic function, deciding whether or not a value is a member of any particular set requires some notion about how the set is constructed, and manifold of the connecting surface (Cox 1994). We can visualise this problem in Figure 2. It is also important that we have the exact knowledge of the sets' characteristics and their elements so we will not waste unnecessary time combining elements with $\mu = 0$.

After the initial forms proposed by Zadeh (minimum μ for intersection and maximum μ for union), several other different ways to deal with intersection and union of fuzzy sets have appeared.

To specify the characteristics of the general concepts that the union and intersection operations must obey a set of axioms, t-norms and t-conorms were created and named respectively.

The t-norms (T) generalise the concept of intersection that must satisfy the following axioms:

$x \, T \, 0 = 0, \forall x \in [0,1]$
$x \, T \, 1 = x, \forall x \in [0,1]$
$x \, T \, y = y \, T \, x$
$x \, T \, (y \, T \, z) = (x \, T \, y) \, T \, z$
$z \, T \, w \leq x \, T \, y$ if $z \leq x$ and $w \leq y$

The algebraic product is another example of possible intersection of sets. The algebraic product "AB" for the sets "A" and "B" respectively with elements "x" and "y" is defined as:

$$\mu AB\ (x) = \mu A\ (x) * \mu B\ (y), \forall x = y \quad (5)$$

The t-conorms ($\perp$) generalize the concept of union that must satisfy the following axioms:

$x \perp 0 = x, \forall x \in [0,1]$
$x \perp 1 = 1, \forall x \in [0,1]$
$x \perp y = y \perp x$
$x \perp (y \perp z) = (x \perp y) \perp z$
$z \perp w \leq x \perp y$ if $z \leq x$ and $w \leq y$

The algebraic sum is another example of a possible set's union operation. The algebraic sum "A⊕B" for the sets "A" and "B" respectively with elements "x" and "y" is defined as:

$$\mu A\oplus B\ (x) = \mu A\ (x) + \mu B\ (y) - \mu AB\ (x), \forall x = y \quad (6)$$

It is important to notice that the t-conorms and the t-norms are reduced to the classic union and intersection operators when dealing with crispy sets (conventional sets).