Note

Optimum Trigonometric Ellipsoidal Height, Direct and Inverse

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A new set of algorithms is presented, to compute ellipsoidal height differences by observed vertical angles (trigonometric levelling), and by differential GPS. Numerical examples are included.

Introduction

By using a set of formulas and respective numerical examples, the paper describes the following procedures:

a. Forward solution, by observed vertical at known point A to unknown point B;
b. Reverse solution observed at unknown point A to known point B;
c. Solution by vector components measured by DGPS.

Besides, the paper highlights the validity of trigonometrical height of point B even by picking co-ordinates from a topographic map, and explains that the solution is insensitive to the effect of cartographic inaccuracy.

1. Direct Trigonometric Ellopsoidal Height of the Unknown Point *B*, with Station on Known Trigonometrical Point *A*

To obtain the *direct trigonometric ellipsoidal* height, we start from the following data:

- Trigonometric point A (station) as known point;
- Estimated position B_0 (even cartographic), of unknown point B, of which the ellipsoidal height H_B is required;
- The *spatial distance* [1] $\overline{AB} = d$ (A and B at different height) provided by geodimeter;
- The observed zenith distance ζ_0 of B seen from A.

a. Data of Problem in Rome '40 Co-ordinates

$$\text{Station } A \equiv \begin{cases} \varphi_A = 38°\ 15'\ 20''.\ 142\ N \\ \lambda_A = 15°\ 42'\ 51''.942\ E \\ H_A = 68.924\ m \end{cases}$$

(Scilla lighthouse)

[1] Also known as "Slope Distance".

Estimated position of B (even cartographical) $B_0 \equiv \begin{cases} \varphi_{B_0} = 38° \, 10'.00 \, N \\ \lambda_{B_0} = 15° \, 47'.60 \, E \\ H_{B_0} \cong 500 \, m \end{cases}$

The difference [2] S from the true position, S = 338 m, is equivalent to 1.3cm on IGM (Istituto Geografico Militare) map at scale 1: 25.000.

Respectively, the *Spatial distance* and the *Observed Zenith distance* are as follows:

$\overline{AB} = d = 11727.616m$ and $\zeta_O = 86°38'16''9$

b. Calculation of Rectangular Co-ordinates of A and B_0 (Direct Transformation)

Starting from the following parameters of *Rome'40 Ellipsoid (Hayford International Ellipsoid)*:

$$\begin{cases} a = 6378388m \\ e^2 = 0.00672267 \\ N_A = a(1 - e^2 \sin^2 \varphi_A)^{-\frac{1}{2}} = 6386623.38488 \end{cases}$$

we obtain the following rectangular co-ordinates of A:

$$\begin{cases} X_A = (N_A + H_A)\cos\varphi_A \cos\lambda_A = 4827741.53273 \\ Y_A = (N_A + H_A)\cos\varphi_A \sin\lambda_A = 1358328.8298 \\ Z_A = \left[N_A \left(1 - e^2 \right) + H_A \right] \sin\varphi_A = 3927868.12655 \end{cases}$$

Similarly, the rectangular co-ordinates of B_0 are as follows:

$$\begin{cases} X_{B0} = 4832068.047 \\ Y_{B0} = 1366730.197 \\ Z_{B0} = 3920378.146 \end{cases}$$

[2] At middle latitudes, around 45°, a difference S = 1000m is acceptable. Such a difference is about 4 cm on a IGM map at scale 1:25000.

c. XYZ Co-ordinates of B_1, Adjusted to Agree with Measured Distance d:

$$\begin{cases} \alpha_0 = \dfrac{X_{B0} - X_A}{d_0} = 0.358801036774 \\ \beta_0 = \dfrac{Y_{B0} - Y_A}{d_0} = 0.696731612001 \\ \gamma_0 = \dfrac{Z_{B0} - Z_A}{d_0} = -0.62114964127 \end{cases}$$

with: $d_0 = \left[(X_{B0} - X_A)^2 + (Y_{B0} - Y_A)^2 + (Z_{B0} - Z_A)^2\right]^{\frac{1}{2}} = 12058.25465m$

which replaced in the previous equations yelds for the point B_1:

$$B_1 \begin{cases} X_{B1} = X_A + \alpha_0 d = 4831949.41451 \\ Y_{B1} = Y_A + \beta_0 d = 1366499.8306 \\ Z_{B1} = Z_A + \gamma_0 d = 3920583.52208 \end{cases} \qquad (1)$$

d. Conversion XYZ to Geodetic φ, λ, H for Point B_1 (Vassalio 1997)

Starting from the following values of r_{B1}, φ_0, N_0 and r_0 we obtain φ, λ, H of B_1:

$$r_{B1} = \left(X_{B1}^2 + Y_{B1}^2\right)^{\frac{1}{2}} = 5021459.64152$$

$$\varphi_0 = tg^{-1}\left(\frac{Z_{B1}}{r_{B1}(1-e^2)}\right) = 38°.1691200384$$

$$N_0 = a\left(1 - e^2 \sin^2 \varphi_0\right)^{-\frac{1}{2}} = 6386591.8085$$

$$r_0 = N_0 \cos \varphi_0 = 5021075.0901$$

Therefore, through an inverse transformation, we obtain the following co-ordinates:

$$\varphi_{B1} = tg^{-1}\left(\frac{Z_{B1}}{r_{B1} - r_0 e^2}\right) = 38°.1691056098$$

$$\lambda_{B1} = tg^{-1}\left(\frac{Y_{B1}}{X_{B1}}\right) = 15°.7911724526$$

with the following Great Normal in B_1: $N_{B1} = 6386591.80324$

e. Zenith Distance ζ_V Corrected for Refraction

On the Normal Sphere[3] in φ_A (Vassallo 1987) we calculate the spherical latitude Φ_{B1} of B_1, through Ψ:

$$\Psi = tg^{-1}\left[tg\varphi_{B1}\left(1-e^2\right)\right] = 37°.9814832498$$

Coordinates of B_1 on the Normal Sphere

$$\begin{cases} \underline{\Phi_{B1}} = \Psi + \sin^{-1}\left(\cos\Psi \cdot e^2 \cdot \sin\varphi_A\right) = \underline{38°.1694657059} \\ \underline{\lambda_{B1}} = \lambda_{B1} = \underline{15°.7911724526} \end{cases}$$

We now proceed with the rigorous reduction to the Local Sphere (Normal Sphere) of the distance

$d = \overline{AB} = \overline{AB_1}$. So that, let us consider for the point A the following expressions:

$$\begin{cases} x_A = \cos\varphi_A \cos\lambda_A = 0.755906390516 \\ y_A = \cos\varphi_A \sin\lambda_A = 0.212681112039 \\ z_A = \sin\varphi_A = 0.619170633476 \end{cases}$$

and for the point B_1 the following ones:

$$\begin{cases} x_{B1} = \cos\Phi_{B1} \cos\lambda_{B1} = 0.75651562026 \\ y_{B1} = \cos\Phi_{B1} \sin\lambda_{B1} = 0.213946458923 \\ z_{B1} = \sin\Phi_{B1} = 0.617989505588 \end{cases}$$

Given σ as angle at centre of the Normal Sphere and subtended by the distance $d = \overline{AB} = \overline{AB_1}$, we can write the following equation:

$$\underline{\sigma} = \cos^{-1}\left(x_A x_{B1} + y_A y_{B1} + z_A z_{B1}\right) = \underline{0°.105139378175}$$

and considering σ in radians (σ_r), the following spherical geodetic distance S is obtained:

$S = \sigma_r \cdot N_A$ with N_A = radius of the Normal Sphere in φ_A

For hot weather, we assume a *coefficient* of vertical *refraction* k = 0.12. It follows that the'*True or corrected Zenith Distance*' is given by the following expression, using the observed Zenith Distance ζ_o :

$$\zeta_v = \zeta_o + \frac{\sigma}{2} \cdot k = 86°.6443361405$$

f. Calculation of Ellipsoidal Height for Point B

From the following value of R_A (as distance from point A and the centre of the Normal Sphere):

$$\underline{R_A} = N_A + H_A = \underline{6386692.30888}$$

[3] Sphere with Great Normal $N = a\left(1-e^2\sin^2\varphi_A\right)^{-\frac{1}{2}}$ as radius.

through the following intermediate steps:

$$\underline{Z_{B11}} = N_{B1}\left(1-e^2\right)\sin\varphi_{B1} = \underline{3920282.01835}$$

$$\underline{r_{B11}} = N_{B1}\cos\varphi_{B1} = \underline{5021076.07988}$$

$$\underline{R_{B11}} = \left[\left(Z_{B11} + N_A e^2 \sin\varphi_A\right)^2 + r_{B11}^2\right]^{\frac{1}{2}} = \underline{6386623.35473}$$

and using 'Carnot formula' for R:

$$\underline{R} = \left(R_A^2 + d^2 + 2dR_A\cos\zeta_v\right)^{\frac{1}{2}} = \underline{6387389.50147}$$

Thus we arrive at the desired '*Ellipsoidal height of B*':

$$H_B = R - R_{B11} = 766.147m$$

At figure 1 the Normal Sphere and the different symbols mentioned in the paper are shown. All the points are contained in the distinct planes APO_A, ABO_A, AB_1O_A, PBO_B, e PB_1O_{B1}.

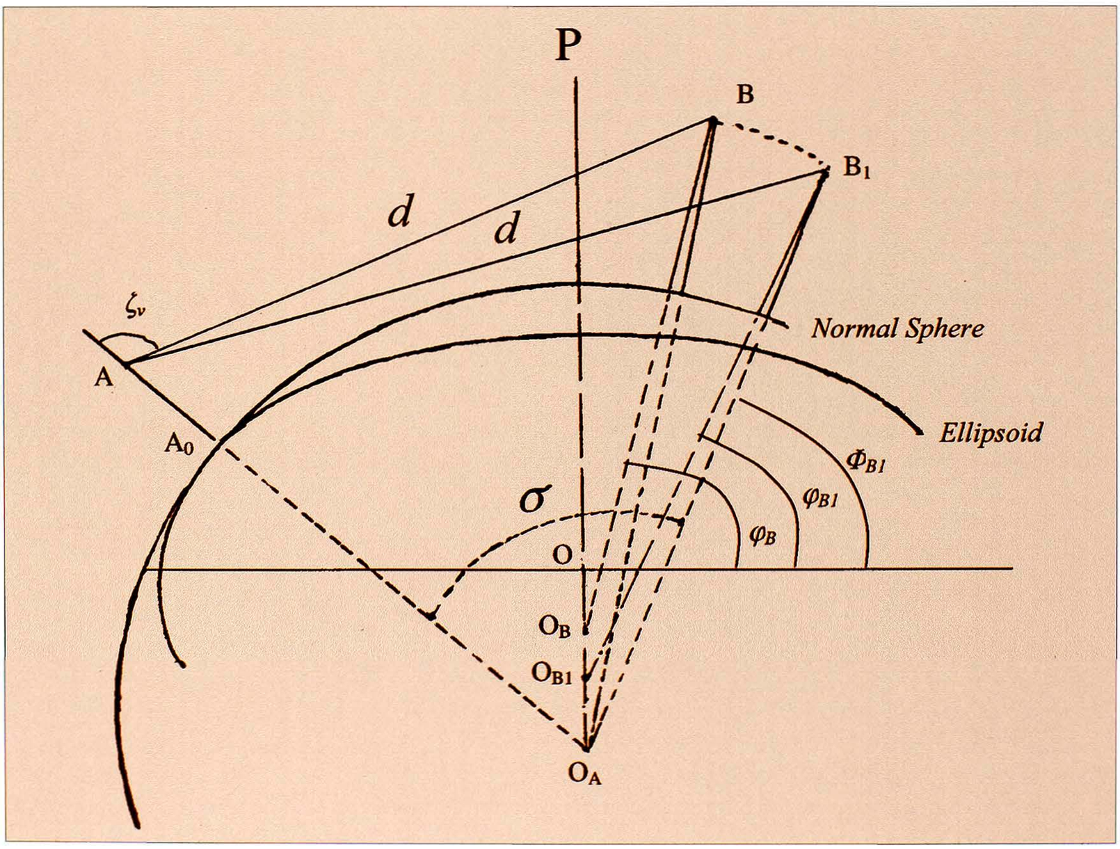

Figure 1: Normal sphere and ellipsoid

2. 'Inverse' Trigonometric Ellipsoidal Height, or Height of the Unknown Station A from Which the Known Trigonometric Point B Is Observed

The two starting points of the problem, both in Rome'40 co-ordinates, are the *Estimated Station Point (even cartographic)* A_0, the difference S from the true position of which is S = *225m*, that is equivalent to *9mm ~ 1cm* on an IGM map at scale 1:25.000, and the *known Trigonometric Point (S. Stefano in Aspromonte)*. The co-ordinates of the two points are respectively as follows:

$$A_o \equiv \begin{cases} \varphi_{A0} = 38°15'15''N \\ \lambda_{A0} = 15°42'45''E \\ H_{A0} \cong 40m \end{cases} \qquad B \equiv \begin{cases} \varphi_B = 38°10'10''.0889N \\ \lambda_B = 15°47'29''.6107E \\ H_B = 766.148m \end{cases}$$

whilst the *Spatial Distance* and the *Zenith Distance of B observed from A*, are respectively as follows:

$$d = \overline{AB} = 11727.616m \quad \text{and} \quad \zeta_O = 86°.6380277778$$

a. Calculation of Rectangular Co-ordinates of B and A_0

Using already known formulas, we obtain the following values:

$$N_B = 6386591.93593m \qquad N_{A0} = 6386622.86313m$$

$$B \equiv \begin{cases} X_B = 4832126.727 \\ Y_B = 1366585.1398 \\ Z_B = 3920787.21463 \end{cases} \qquad A_0 \equiv \begin{cases} X_{A0} = 4827859.88032 \\ Y_{A0} = 1358186.77482 \\ Z_{A0} = 3927725.71738 \end{cases}$$

b. XYZ Co-ordinated of A_1, Adjusted to Agree with Measured Distance d:

Similarly to the calculations made at paragraph 1.c for the point B_1, we obtain for A_1 the following results:

$$\begin{cases} \alpha_0 = \dfrac{X_{A0} - X_B}{d_0} = -0.364699275665 \\ \beta_0 = \dfrac{Y_{A0} - Y_B}{d_0} = -0.717831696281 \\ \gamma_0 = \dfrac{Z_{A0} - Z_B}{d_0} = 0.593053196728 \end{cases}$$

with: $d_0 = \left[(X_{A0} - X_B)^2 + (Y_{A0} - Y_B)^2 + (Z_{A0} - Z_B)^2 \right]^{\frac{1}{2}} = 11699.6296m$

which replaced in the previous equations yields for the point A_1:

$$A_1 \equiv \begin{cases} X_{A1} = X_B + \alpha_0 d = 4827849.67394 \\ Y_{A1} = Y_B + \beta_0 d = 1358166.68531 \\ Z_{A1} = Z_B + \gamma_0 d = 3927742.31479 \end{cases} \quad (2)$$

c. Conversion *XYZ* to Geodetic φ, λ, H for Point A_1 (Vassallo 1997)

Starting from the following values of r_{A1}, φ_0, N_0 e r_0, we obtain φ, λ, H di A_1:

$$r_{A1} = \left(X_{A1}^2 + Y_{A1}^2\right)^{\frac{1}{2}} = 5015251.66061$$

$$\varphi_0 = tg^{-1}\left(\frac{Z_{A1}}{r_{A1}\left(1-e^2\right)}\right) = 38°.2543703563$$

$$N_0 = 6386622.93754$$

$$r_0 = N_0 \cos\varphi_0 = 5015221.51586$$

Therefore, through an inverse transformation, we obtain the following co-ordinates:

$$\begin{cases} \varphi_{A1} = tg^{-1}\left(\dfrac{Z_{A1}}{r_{A1} - r_0 e^2}\right) = 38°.254369223 \\ \lambda_{A1} = tg^{-1}\left(\dfrac{Y_{A1}}{X_{A1}}\right) = 15°.7123106441 \end{cases}$$

with the following Great Normal in A_1: $N_{A1} = 6386622.93712$

d. Calculation of True Zenith Distance ζ_V

Similarly to the paragraph 1.e, on the Normal Sphere in φ_B (Vassallo 1987) we calculate the spherical latitude Φ_{A1} of A_1, through Ψ:

$$\Psi = tg^{-1}\left[tg\varphi_{A1}\left(1-e^2\right)\right] = 38°.0666101327$$

Coordinates of A_1 on the Normal Sphere
$$\begin{cases} \underline{\Phi_{A1}} = \Psi + \sin^{-1}\left(\cos\Psi \cdot e^2 \cdot \sin\varphi_B\right) = \underline{38°.2540161528} \\ \underline{\lambda_{A1}} = \lambda_{A1} = \underline{15°.7123106441} \end{cases}$$

We now proceed with the rigorous reduction to the Local Sphere (Normal Sphere). Let's consider for the point B the following expressions:

$$\begin{cases} x_B = \cos\varphi_B \cos\lambda_B = 0.75651414298 \\ y_B = \cos\varphi_B \sin\lambda_B = 0.213951546458 \\ z_B = \sin\varphi_B = 0.617989552694 \end{cases}$$

and for the point A_I the following ones:

$$\begin{cases} x_{A1} = \cos\Phi_{A1}\cos\lambda_{A1} = 0.755930674923 \\ y_{A1} = \cos\Phi_{A1}\sin\lambda_{A1} = 0.212657793516 \\ z_{A1} = \sin\Phi_{A1} = 0.619148994643 \end{cases}$$

Consequently we obtain the following σ :

$$\underline{\sigma} = \cos^{-1}\left(x_B x_{A1} + y_B y_{A1} + z_B z_{A1}\right) = \underline{0°.105002061689}$$

For hot weather, always assuming a *coefficient of vertical refraction k = 0.12*, it follows that the *'True or corrected Zenith Distance'* is given by the following equation:

$$\zeta_v = \zeta_o + \frac{\sigma}{2}\cdot k = 86°.644279017$$

e. Calculation of Ellipsoidal Height for Station A

From the following value of R_B (as distance from point B and the centre of the Normal Sphere):

$$\underline{R_B} = \left[\left(Z_B + N_{A1}e^2\sin\varphi_{A1}\right)^2 + X_B^2 + Y_B^2\right]^{\frac{1}{2}} = \underline{6387389.05623m}$$

working through the sine law to find σ_0 and R_A:

$$\underline{\sigma_0} = \sin^{-1}\left(\frac{d\sin\zeta_v}{R_B}\right) = \underline{0°.10501805439}$$

$$\underline{R_A} = \frac{\sin(\zeta_v - \sigma_0)R_B}{\sin\zeta_v} = \underline{6386691.86196}$$

we arrive at the desired height H_A:

$$H_A = R_A - N_{A1} = 68.9248m$$

3. Trigonometric Ellipsoidal Height with Two GPS Receivers

In this specific case we don't discuss anymore about direct or inverse height. The two GPS receivers provide, for both A and B points, distance, position and height in WGS'84 system. Practically, in a two hour session, we obtain the following data in WGS'84 system:

- The components of 'Base-line' $\overrightarrow{AB}$ (in metres)

$$\begin{cases} dX_{BA} = X_{BW} - X_{AW} = 4385.224 \\ dY_{BA} = Y_{BW} - Y_{AW} = 8256.322 \\ dZ_{BA} = Z_{BW} - Z_{AW} = -7080.955 \end{cases}$$

and inversely the components of 'Base-line' $\overrightarrow{BA}$ (in metres):

$$\begin{cases} dX_{AB} = X_{AW} - X_{BW} = -dX_{BA} \\ dY_{AB} = Y_{AW} - Y_{BW} = -dY_{BA} \\ dZ_{AB} = Z_{AW} - Z_{BW} = -dZ_{BA} \end{cases}$$

- The 'Slope-distance' d:

$$d = \left(dX^2 + dY^2 + dZ^2\right)^{\frac{1}{2}} = 11727.616m$$

a. Vectorial Solution

The vectorial solution is a '*hybrid*' solution that utilises the dX, dY, dZ as if were Rome'40 components. The results coming from such a solution are very good, and permit to disregard the rotary elements of the two systems (Rome'40 and WGS'84).
We start now to resolve the problem at paragraph 1, in order to determine the height of B. We can write as follows:

$$\begin{cases} X_B = X_A + dX_{BA} = 4832126.75673 \\ Y_B = Y_A + dY_{BA} = 1366585.1518 \\ Z_B = Z_A + dZ_{BA} = 3920787.17155 \end{cases} \qquad (3)$$

From the formulas (3), using the inverse transformations through r_B, φ_0, N_0 and r_0:

$$r_B = \left(X_B^2 + Y_B^2\right)^{\frac{1}{2}} = 5021653.50957$$

$$\varphi_0 = tg^{-1}\left(\frac{Z_B}{r_B\left(1-e^2\right)}\right) = 38°.1694912543$$

$$N_0 = 6386591.944$$

$$r_0 = N_0 \cos\varphi_0 = 5021049.62534$$

we obtain, through inverse transformation, the following co-ordinates φ and λ of B in Rome'40:

$$\begin{cases} \varphi_B = tg^{-1}\left(\dfrac{Z_B}{r_B - r_0 e^2}\right) = 38°10'10''.086949 \text{ N} \\ \lambda_B = tg^{-1}\left(\dfrac{Y_B}{X_B}\right) = 15°47'29''.610 \text{E} \end{cases}$$

with the following Great Normal in B:

$$N_B = 6386591.9357$$

We arrive at the desired height H_B:

$$H_B = \frac{r_B}{\cos\varphi_B} - N_B = 766.142m$$

Resolving now the problem at paragraph 2, we obtain the height of A. We can write as follows:

$$\begin{cases} X_A = X_B + dX_{AB} = 4827741.503 \\ Y_A = Y_B + dY_{AB} = 1358328.8178 \\ Z_A = Z_B + dZ_{AB} = 3927868.16963 \end{cases} \quad (4)$$

From the formulas (4), using again the known inverse transformations, and going through r_A, φ_0, N_0 and r_0:

$$r_A = 5015191.44172 \qquad \varphi_0 = 38°.2555974587$$

$$N_0 = 6386623.38577 \qquad r_0 = 5015137.17749$$

we obtain, through inverse transformation, the following co-ordinates of A in Rome'40:

$$\begin{cases} \varphi_A = 38°15'20''.1435\,\mathrm{N} \\ \lambda_A = 15°42'51''.94212\,\mathrm{E} \end{cases}$$

with the following Great Normal in A:

$$N_A = 6386623.38503$$

we arrive at the desired height H_A:

$$H_A = 68.925m$$

Conclusion

We have presented algorithms to compute ellipsoidal height differences, between stations which are linked by observed vertical angles.

In order that the calculated height is *ellipsoidal*, it is necessary that the known height of the known point (point A in the first discussed case and point B in the second discussed case) is ellipsoidal. It is important to underline that the *height of the point of the Datum* is ellipsoidal, because the datum is the point of emanation of the National Trigonometric Network. Such a height is obtained through Geometric Levelling. This levelling starts from the Mean Sea Level (level of zero height), provided in Italy by a Tide-gauge in Genoa, and reaches the fundamental point of Datum. The height of Datum is assumed as ellipsoidal height, and therefore if the known point or the supporting point is the point of the Datum, then, using the above process, we obtain all *Trigonometric Ellipsoidal* heights. Consequently, starting from the Datum and proceeding to determine trigonometric heights of the Italian Network, if we use the above methodology we will obtain all ellipsoidal heights.

The above methodology allows also to obtain the ellipsoidal height of point B inclusive of difference between the ellipsoid and the mean Sea Level in point A.

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