The Three-point Problem of The Median Line Turning Point: on the Solutions for the Sphere and Ellipsoid

Lars E. Sjöberg, Royal Institute of Technology, Division of Geodesy, Sweden

The problem of determining turning points of median lines between states separated by sea is considered. The turning point is defined as the point with equidistance lines to two basepoints along the shoreline of one state and one basepoint in the adjacent state. For the sphere the equidistance lines are parts of great circles, and the problem is solved by closed formulas. For the ellipsoid the lines are defined along geodesics, and an iterative solution is presented.

Introduction

The median line is a line *every point of which is equidistant from nearest points on the baselines of two states* (IHB, 1993). This line is crucial for the maritime delimitation between opposite coasts of two states separated by sea. In general, the median line runs smoothly as the 'straight line' equidistant between two distinct *baselines*, each belonging to one of the states. However, whenever points belonging to another baseline along one coastline get closer to the median line than points on the previous baseline, the median line turns. As suggested by Carrera (1987) the median line turning point can be defined by a *three-point method*, i.e. the turning point is the point equidistant to three baseline points (belonging to different baselines). If the scale of the baselines (between defined *basepoints*) is small compared to the scale of the coastline separation, the points along baselines can be approximated by the discrete basepoints. This approximation will be used here to determine median line turning points. This problem was also treated by Carrera (1987), Horemuz (1999), Horemuz et al.(1999) and Fan (2001). Carrera(1987) solved the problem by classical formulas for solving the geodetic direct and indirect problems for geodesics on the ellipsoid. Horemuz (1999) and Horemuz et al. (1999) solved the problem explicitly by rectangular co-ordinates for the spherical surface of reference, while an approximate method was used for the ellipsoidal surface of reference.

In this paper the three-point turning point problem is solved explicitly for the sphere by spherical co-ordinates. For an ellipsoidal surface of reference the problem is formulated by integral equations along geodesics.

Solutions for the Sphere

Proposition: Given the points $P_i(\varphi_i, \lambda_i)$ with latitudes φ_i and longitudes λ_i; i = 1,2,3, the three-point problem has the solutions

Case I : ($\varphi_1 \neq \varphi_2$):

$$\tan\lambda = \frac{S_{23}\cos\varphi_1\cos\lambda_1 - S_{13}\cos\varphi_2\cos\lambda_2 + S_{12}\cos\varphi_3\cos\lambda_3}{-S_{23}\cos\varphi_1\sin\lambda_1 + S_{13}\cos\varphi_2\sin\lambda_2 - S_{12}\cos\varphi_3\sin\lambda_3} \quad (1a)$$

and

$$\tan\varphi = \frac{\cos\varphi_2\cos\Delta\lambda_2 - \cos\varphi_1\cos\Delta\lambda_1}{S_{12}}, \quad (1b)$$

where

$$S_{ij} = \sin\varphi_i - \sin\varphi_j \text{ ; i,j =1,2,3} \quad \text{and} \quad \Delta\lambda_i = \lambda - \lambda_i \text{ ; i =1,2.}$$

Case II : ($\varphi_1 = \varphi_2 \neq \varphi_3$):

$$\tan\lambda = \frac{\cos\lambda_1 - \cos\lambda_2}{\sin\lambda_2 - \sin\lambda_1} \quad (2a)$$

and

$$\tan\varphi = \frac{\cos\varphi_3\cos\Delta\lambda_3 - \cos\varphi_1\cos\Delta\lambda_1}{S_{13}} \quad (2b)$$

Proof: The point $P(\varphi,\lambda)$ is defined by equal geocentric angles to the known points P_i (φ_i,λ_i);i=1,2,3. From the spherical cosine theorem one obtains the following relation between ψ_i and the spherical co-ordinates of the points P and P_i :

$$\cos\psi_i = \sin\varphi\sin\varphi_i + \cos\varphi\cos\varphi_i\cos\Delta\lambda_i;\ \text{i =1,2,3.} \quad (3)$$

As P (φ,λ) is defined by $\psi_1=\psi_2=\psi_3$, it follows that

$$\sin\varphi\sin\varphi_1 + \cos\varphi\cos\varphi_1\cos\Delta\lambda_1 = \sin\varphi\sin\varphi_2 + \cos\varphi\cos\varphi_2\cos\Delta\lambda_2 = \quad (4)$$
$$= \sin\varphi\sin\varphi_3 + \cos\varphi\cos\varphi_3\cos\Delta\lambda_3$$

or

$$\tan\varphi\sin\varphi_1 + \cos\varphi_1\cos\Delta\lambda_1 = \tan\varphi\sin\varphi_2 + \cos\varphi_2\cos\Delta\lambda_2 = \quad (5)$$
$$= \tan\varphi\sin\varphi_3 + \cos\varphi_3\cos\Delta\lambda_3$$

The first equation of formula (5) (including points P_1 and P_2) can be rewritten:

$$\tan\varphi(\sin\varphi_1 - \sin\varphi_2) = \cos\varphi_2\cos\Delta\lambda_2 - \cos\varphi_1\cos\Delta\lambda_1 \quad (6a)$$

Same treatment of the second equation of formula (5) yields:

$$\tan\varphi(\sin\varphi_1 - \sin\varphi_3) = \cos\varphi_3\cos\Delta\lambda_3 - \cos\varphi_1\cos\Delta\lambda_1 \quad (6b)$$

Let us now consider the different cases of the proposition.

Case I ($\varphi_1 \neq \varphi_2$): In this case φ is eliminated by dividing each member of Eq. (6a) by (6b). For $\varphi_1 \neq \varphi_3$ the result is

$$\frac{S_{12}}{S_{13}} = \frac{\sin\varphi_1 - \sin\varphi_2}{\sin\varphi_1 - \sin\varphi_3} = \frac{\cos\varphi_2\cos\Delta\lambda_2 - \cos\varphi_1\cos\Delta\lambda_1}{\cos\varphi_3\cos\Delta\lambda_3 - \cos\varphi_1\cos\Delta\lambda_1} \quad (7)$$

Inserting into (7)

$$\cos\Delta\lambda_i = \cos\lambda\cos\lambda_i + \sin\lambda\sin\lambda_i \quad (8)$$

one easily arrives at the solution (1a) for λ, and (1b) is directly obtained from Eq.(6a). If $\varphi_1 = \varphi_3$, the left hand side of (6b) vanishes. Together with (8) it yields

$$\tan\lambda = -\frac{\sin\lambda_1 - \sin\lambda_3}{\cos\lambda_1 - \cos\lambda_3} \quad (9)$$

which formula agrees with (1a).

Case II ($\varphi_1 = \varphi_2 \neq \varphi_3$): Formula (8) inserted into Eq. (6a) with $S_{12}= 0$ yields

$$\cos\lambda\cos\lambda_2 + \sin\lambda\sin\lambda_2 = \cos\lambda\cos\lambda_1 + \sin\lambda\sin\lambda_1$$

which can be rewritten on the form (2a). The solution (2b) follows directly from (6b). Q.E.D.
The solution for Case I was also derived by Fan (2001).
Although the above solutions of the proposition are mathematically exact, they may suffer from numerical instability in the practical application. One improvement is gained by substituting the differences of sines of S_{ij} by

$$S_{ij} = 2\cos\varphi_{ij}\sin(\Delta\varphi_{ij}/2) \tag{10}$$

where $\varphi_{ij} = (\varphi_i + \varphi_j)/2$.and $\Delta\varphi_{ij} = \varphi_i - \varphi_j$ A similar improvement can be achieved for the differences of cosines of (9). To avoid the division by near zero for small latitude and/or longitude differences among the known points the following corollary may be useful.
Corollary 1: The coordinates λ and φ of the median line turning point is given by

$$\sin\lambda = \pm\frac{A_1}{D_1}, \quad \cos\lambda = \pm\frac{B_1}{D_1} \tag{11}$$

and

$$\sin\varphi = \pm\frac{A_2}{D_2}, \quad \cos\varphi = \pm\frac{B_2}{D_2} \tag{12}$$

where A_1 and B_1 are the numerator and denominator, respectively, of the right hand side of Eq.(1a), and

$$D_1 = \sqrt{A_1^2 + B_1^2} = \sqrt{S_{23}^2\cos^2\varphi_1 + S_{13}^2\cos^2\varphi_2 + S_{12}^2\cos^2\varphi_3} \tag{13}$$

$$A_2 = \cos\varphi_2\cos\Delta\lambda_2 - 2\cos\varphi_1\cos\Delta\lambda_1 + \cos\varphi_3\cos\Delta\lambda_3 \tag{14a}$$

$$B_2 = S_{12} + S_{13} \tag{14b}$$

$$D_2 = \sqrt{A_2^2 + B_2^2} \tag{14c}$$

The proof is given by simple trigonometric manipulations of Eqs. (1a) and (6a,b).
Unfortunately, there is still the possibility that for small baselines the denominators D1 and D2 may be small, making the solutions for λ and φ numerically unstable. This problem must be further studied.
Corollary 2: The azimuth α_i at point P_i (φ_i,λ_i) along the great circle towards P (φ,λ) is given by

$$\tan\alpha_i = \frac{\sin(\lambda - \lambda_i)}{\cos\varphi_i\tan\varphi - \sin\varphi_i\cos(\lambda - \lambda_i)}; \quad i = 1,2,3 \tag{15}$$

The proof is given e.g. in Sjöberg (2002).

Solution for the Ellipsoid

In the case of an ellipsoidal surface of reference it is convenient to introduce the reduced latitude β related by the geodetic latitude φ by the relation

$$\tan\beta = \sqrt{1-e^2}\tan\varphi \tag{16}$$

where $e = \sqrt{a^2 - b^2}/a$ is the first excentricity of the ellipsoid defined by the semi-major and -minor axes a and b. The distance (s_i) and the ellipsoidal longitude difference (ΔL_i) from the point P_i (β_i, L_i) to the wanted turning point P (β, L) along the geodesic are given by

$$s_i = \int_{\beta_i}^{\beta} f(\beta, h_i)\,d\beta = F(\beta, h_i) - F(\beta_i, h_i) \tag{17a}$$

and

$$\Delta L_i = L - L_i = \int_{\beta_i}^{\beta} g(\beta, h_i)\,d\beta = G(\beta, h_i) - G(\beta_i, h_i) \tag{18a}$$

where

$$f(\beta, h_i) = a\sqrt{\frac{1-e^2\cos^2\beta}{\cos^2\beta - h_i^2}}\cos\beta \tag{17b}$$

and

$$g(\beta, h_i) = \pm\frac{h_i}{\cos\beta}\sqrt{\frac{1-e^2\cos^2\beta}{\cos^2\beta - h_i^2}} \tag{18b}$$

Here $h_i = \cos\beta_{MAX}$, where β_{MAX} is the maximum (or minimum) latitude of the geodesic.
Alternatively, using the variable substitutions

$$\sin v = \frac{\sin\beta}{\sqrt{1-h^2}} \tag{19a}$$

and

$$\sin w = \frac{h}{\sqrt{1-h^2}}\tan\beta \tag{19b}$$

the integrals (17a) and (18a) can also be written (Klotz 1991 and 1993; Schmidt 1999 and 2000):

$$s_i = \int_{v_i}^{v} f^*(v, h_i)\,dv = F^*(v, h_i) - F^*(v_i, h_i) \tag{20a}$$

and

$$\Delta L_i = \int_{w_i}^{w} g^*(w, h_i)\,dw = G^*(w, h_i) - G^*(w_i, h_i) \tag{21a}$$

where

$$f^*(v, h_i) = a\sqrt{1-e^2\left\{1-\left(1-h_i^2\right)\sin^2 v\right\}} \tag{20b}$$

and

$$g^*(w, h_i) = \pm\sqrt{1-\frac{e^2h_i^2}{h_i^2+\left(1-h_i^2\right)\sin^2 w}} \tag{21b}$$

The three-point problem can now be defined by the equations

$$s_1 = s_2 = s_3 \tag{22a}$$

and

$$L = L_1 + \Delta L_1 = L_2 + \Delta L_2 = L_3 + \Delta L_3 \tag{22b}$$

where s_i and ΔL_i are given by Eqs. (17a) and (18a) or by Eqs.(20a) and (21a). The longitude L can be eliminated from (22b), yielding four independent equations with four unknowns (β, h_1, h_2, h_3). The system of equations can thus be written

$$
\begin{aligned}
s_1 &= s_2 \\
s_1 &= s_3 \\
L_1 + \Delta L_1 &= L_2 + \Delta L_2 \\
L_1 + \Delta L_1 &= L_3 + \Delta L_3
\end{aligned}
\qquad (23)
$$

Inserting formulas (17a) and (18a) and linearizing, one obtains the matrix equation

$$\mathbf{AX} = \mathbf{Y} \qquad (24a)$$

where

$$
\begin{aligned}
\mathbf{X}^T &= (\Delta\beta, \Delta h_1, \Delta h_2, \Delta h_3) \\
\mathbf{Y}^T &= \{(s_2)-(s_1),(s_3)-(s_1),L_2-(L_2)-L_1+(L_1),L_3-(L_3)-L_1+(L_1)\}
\end{aligned}
\qquad (24b)
$$

Here the bracket ( ) denotes an approximation to the quantity within the bracket, determined by β^0 and H_1^0. The vector of unknowns **X** contains improvements to the approximate values β^0, h_1^0, h_2^0 and h_3^0. The elements of the (4x4) design matrix **A** are presented in the Appendix. In order to determine the elements of **Y** and **A** the integrals (17a) and (18a) or (20a) and (21a) must be employed, e.g. by series expansions or direct numerical integrations (Klotz 1991 and 1993; Schmidt 1999 and 2000). Starting values for β^0, v^0 and h_i^0 are preferably given by the spherical solutions of Section 2. As the equations are linearised, the solution should be iterated.

Concluding Remarks

The solution of the position of a median line turning point from the three-point problem was derived explicitly for the sphere and as an iterative vector solution for the ellipsoid. The problem with possible unstable solutions for small baselines deserves further attention. However, numerical examples are left for a forthcoming paper. A future challenge is to avoid the approximation by the three-point problem and to determine the position of the turning point directly from all points along the baselines.

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Appendix

The elements of the design matrix **A** are as follows:

$$A_{11}=\left(\frac{\partial s_1}{\partial \beta}\right)-\left(\frac{\partial s_2}{\partial \beta}\right)=f\left(\beta^0,h_1^0\right)-f\left(\beta^0,h_2^0\right)$$

$$A_{12}=\left(\frac{\partial s_1}{\partial h_1}\right)=\int_{\beta_1}^{\beta^0}\left\{\frac{\partial f}{\partial h_1}\left(\beta,h_1^0\right)\cos\beta d\beta\right\}=\frac{ah_1^0}{1-\left(h_1^0\right)^2}\int_{v_1}^{v^0}\frac{f^*\left(v,h_1^0\right)}{\cos^2 v}dv$$

$$A_{13}=-\left(\frac{\partial s_2}{\partial h_2}\right)=\frac{-ah_2^0}{1-\left(h_2^0\right)^2}\int_{v_2}^{v^0}\frac{f^*\left(v,h_2^0\right)}{\cos^2 v}dv$$

$$A_{14}=0$$

$$A_{21}=\left(\frac{\partial s_1}{\partial \beta}\right)-\left(\frac{\partial s_3}{\partial \beta}\right)=f\left(\beta^0,h_1^0\right)-f\left(\beta^0,h_3^0\right)$$

$$A_{22}=A_{12}$$

$$A_{23}=0$$

$$A_{24}=-\left(\frac{\partial s_3}{\partial h_3}\right)=\frac{-ah_3^0}{1-\left(h_3^0\right)^2}\int_{v_3}^{v^0}\frac{f^*\left(v,h_3^0\right)}{\cos^2 v}dv$$

$$A_{31}=\left(\frac{\partial \Delta L_1}{\partial \beta}\right)-\left(\frac{\partial \Delta L_2}{\partial \beta}\right)=g\left(\beta^0,h_1^0\right)-g\left(\beta^0,h_2^0\right)$$

$$A_{32}=\left(\frac{\partial \Delta L_1}{\partial h_1}\right)=\int_{\beta_1}^{\beta^0}\frac{\partial g}{\partial h_1}d\beta=\frac{h_1^0}{1-\left(h_1^0\right)^2}\int_{v_1}^{v^0}\frac{g^*\left(v,h_1^0\right)}{\cos^2 v}dv$$

$$A_{33}=-\left(\frac{\partial \Delta L_2}{\partial h_2}\right)=-\frac{h_2^0}{1-\left(h_2^0\right)^2}\int_{v_2}^{v^0}\frac{g^*\left(v,h_2^0\right)}{\cos^2 v}dv$$

$$A_{34}=0$$

$$A_{41}=\left(\frac{\partial \Delta L_1}{\partial h_1}\right)-\left(\frac{\partial \Delta L_3}{\partial h_3}\right)=\frac{h_1^0}{1-\left(h_1^0\right)^2}\int_{v_1}^{v^0}\frac{g^*\left(v,h_1^0\right)}{\cos^2 v}dv-\frac{h_3^0}{1-\left(h_3^0\right)^2}\int_{v_3}^{v^0}\frac{g^*\left(v,h_3^0\right)}{\cos^2 v}dv$$

$$A_{42}=A_{32}$$

$$A_{43}=0$$

$$A_{44}=-\left(\frac{\partial \Delta L_3}{\partial h_3}\right)=-\frac{h_3^0}{1-\left(h_3^0\right)^2}\int_{v_3}^{v^0}\frac{g^*\left(v,h_3^0\right)}{\cos^2 v}dv$$

Biography

Lars E. Sjöberg got his Ph.D. in Geodesy in 1975 at the Royal Institute of Technology (KTH) in Stockholm. After working with professor R.H Rapp at The Ohio State University during 1977 and 1978 and about four years work with the National Land Survey of Sweden, he returned to KTH in 1984 to succeed his old professor on the chair of Geodesy.

Through the years he has been a member of several IAG special study groups (chairing three of them) and commissions, and he is an IAG Fellow since 1991. He is a presidium member of the Nordic Geodetic Commission since 1982, a Fellow of the Alexander-von-Humboldt Foundation since 1983 and a corresponding member of the German Geodetic Commission since 1989.

His research interests are in the fields of geodetic theory of errors, physical geodesy, GPS positioning and deformation analysis. From 1989 to 1995 he served in the editorial boards of Manuscripta Geodaetica and Bulletine Geodesique, and he has published about 200 scientific papers and reports.

E-mail: sjoberg@geomatics.kth.se