Road Network Design: Optimal Economic Connection of Three Horizontal Control Points on Flat, Uniform Terrain

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ABSTRACT

Road network design may involve selection of junction points, and a standard economic criterion can guide the initial paper location of these points. The junction point problem, in its elemental form, begins with three horizontal control points. These control points are to be linked at minimum total cost using linear road segments. Each road segment is characterized by a constant cost per unit length which may differ between segments. A mathematical optimization model has been developed for this problem. A decision algorithm for optimal network design is given and an example from the literature illustrates its application.

Keywords: *harvest planning, route location, forest access, road networks, optimization, logging engineering.*

INTRODUCTION

Optimal economic design of a road network connecting three control points is a fundamental transportation problem. It is the purpose of this paper to present an easily implemented optimization algorithm for this problem. The paper begins by acknowledging the formative work done on this topic by Professor Wilhelm Launhardt. His early contribution toward the analytical analysis of this particular problem is indicated during the course of solution development. The definitive analytical solution presented in this paper provides the core of a comprehensive optimization algorithm which is presented and applied. Some observations with regard to potential utilization of the algorithm conclude the paper.

The author is a Professor in the Department of Forest Management and Engineering.

BACKGROUND

Starting in the last half of the 19th century, Professor Wilhelm Launhardt of the Polytechnical College of Hannover, Germany, did pioneering work in the economics of transportation. An English-language publication on road location contains some of his major contributions to this topic [6]. Professor Launhardt made an essential distinction between the "commercial trace" and the "technical trace" of a road. (The term trace referred to the plan view of a proposed road location.) The commercial trace is defined to be the optimal economic location of the road under the assumption of "perfectly horizontal and uniform ground." Subsequent modification of this line done in response to more detailed observation of on-the-ground conditions results in the technical trace. This stepped process, albeit more detailed, continues to be an essential feature of transportation system planning and design [9,10].

In the development of guiding principles for optimal economic location of roads, Launhardt describes "The Principle of the Node." By this principle the total cost of constructing and using the road system between three nodes is to be minimized. In order to apply this principle in practice Launhardt gives two different solution procedures; one is based on geometry, the other on mechanics. Using calculus, he also derives certain mathematical conditions that an interior junction point location must satisfy in order to be the economic optimum. In general, however, these mathematical equations are not of a form suitable for the direct and complete analytical solution of the problem. For this reason the solutions he presents for examples contained in his book were arrived at from careful scaling of either mechanical analogues or geometric constructions.

Launhardt's Principle of the Node may be succinctly stated in mathematical terms; viz., equations 1 and 2 of Table 1 with variables as defined in Table 2. The indicated minimization of this cost function via the calculus leads, after some trigonometric substitution and algebraic manipulation, to a complete closed-form solution for the optimal location of an interior junction point. Launhardt's contribution to the derivation of this analytical solution ends with equations 10 of Table 1. Continued development of this prematurely terminated line of investigation leads to the conclusive closed-form solution. It is this solution, now presented, which is of direct utility in the final algorithm.

[1]	$\underset{(x,y)}{\mathrm{MIN}}\ Z(x,y) = \sum_{i=1}^{3} k_i S_i$
[2a,b,c]	$S_i = [(x_i - x)^2 + (y_i - y)^2]^{1/2}$ for i=1,2,3
[3]	$A = \left[\frac{1}{2}\right][x_1y_2 + x_2y_3 + x_3y_1 - x_2y_1 - x_3y_2 - x_1y_3]$
[4a]	$D_1 = [(x_2 - x_3)^2 + (y_2 - y_3)^2]^{1/2}$
[4b]	$D_2 = [(x_3 - x_1)^2 + (y_3 - y_1)^2]^{1/2}$
[4c]	$D_3 = [(x_1 - x_2)^2 + (y_1 - y_2)^2]^{1/2}$
[5a]	$K_1 = k_1^2 - k_2^2 - k_3^2$
[6a]	$J_1 = D_1^2 - D_2^2 - D_3^2$
[5b]	$K_2 = k_2^2 - k_1^2 - k_3^2$
[6b]	$J_2 = D_2^2 - D_1^2 - D_3^2$
[5c]	$K_3 = k_3^2 - k_1^2 - k_2^2$
[6c]	$J_3 = D_3^2 - D_1^2 - D_2^2$
[7]	$A_L^2 = \left[\frac{1}{16}\right][J_1J_2 + J_2J_3 + J_1J_3]$
[8]	$A_C^2 = \left[\frac{1}{16}\right][K_1K_2 + K_2K_3 + K_1K_3]$
[9a,b,c]	$G_i = K_iA_L + J_iA_C$ for i=1,2,3

Table 1. Listing of formulas used in the paper.

The following equations, [10] through [13], are only applicable for case D of figure 1. Equations 10 are directly based on Launhardt [6].

[10a] $$^{*}a_1 = \arccos\left(\frac{K_1}{2K_2K_3}\right)$$

[10b] $$^{*}a_2 = \arccos\left(\frac{K_2}{2K_1K_3}\right)$$

[10c] $$^{*}a_3 = \arccos\left(\frac{K_3}{2K_1K_2}\right)$$

The following equations were developed by the author. They should be solved sequentially as a continuation of the calculations begun with equations 4 through 9.

[11] $$^{*}Z = \left[\frac{1}{\sqrt{2}}\right]\left[(16A_LA_C)-(K_1D_1^2+K_2D_2^2+K_3D_3^2)\right]^{\frac{1}{2}}$$

[12a,b,c] $$^{*}S_i = \left[\frac{-k_iG_i}{2\ {}^{*}Z\ A_C}\right] \qquad \text{for } i=1,2,3$$

[13a] $$^{*}x = x_1+\left[\frac{D_3^2+{}^{*}S_1^2-{}^{*}S_2^2}{2D_3^2}\right](x_2-x_1)-\left[\frac{2\ {}^{*}S_1{}^{*}S_2\ A_C}{k_1k_2D_3^2}\right](y_2-y_1)$$

[13b] $$^{*}y = y_1+\left[\frac{D_3^2+{}^{*}S_1^2-{}^{*}S_2^2}{2D_3^2}\right](y_2-y_1)+\left[\frac{2\ {}^{*}S_1{}^{*}S_2\ A_C}{k_1k_2D_3^2}\right](x_2-x_1)$$

Table 1 (cont.). Listing of formulas used in the paper.

(x,y)	Coordinate location of the junction point.
$Z(x,y)$	Total cost of constructing and using a road network connecting three given control points when the network has a junction point at (x,y).
(x_i,y_i)	Coordinate location of control point i.
k_i	Cost per unit length of constructing and using the road segment from control point i to the junction point.
S_i	Length of the road segment from control point i to the junction point.
a_i	The junction point angle opposite location triangle side D_i.
$\bar{a}_i$	The angle opposite side k_i of the cost triangle and also the supplement of the optimal junction point angle ${}^*a_i$.
D_i	Length of the location triangle side opposite control point i.
A	Signed area of the location triangle as calculated by the coordinate area formula.
K_i	A convenient intermediate variable relating cost triangle side lengths.
J_i	A convenient intermediate variable relating location triangle side lengths.
A_C^2	Squared area of the cost triangle.
A_L^2	Squared area of the location triangle.
G_i	A convenient intermediate variable relating the cost and location triangles.

Optimal values for a variable are indicated by an asterisk; e.g., $({}^*x,\ {}^*y)$, ${}^*Z$, and ${}^*S_i$.

Table 2. Definitions for variables used in the paper.

JUNCTION POINT ALGORITHM

Following Launhardt [6] there are four possible ways of connecting three non-collinear points with non-redundant pathways (Fig. 1). Only the last configuration shown in Figure 1 requires the creation of an interior junction point; i.e., one that is distinctly separate from the control points. It is this latter case that is of particular interest although each of the other three must also be included in the solution algorithm.

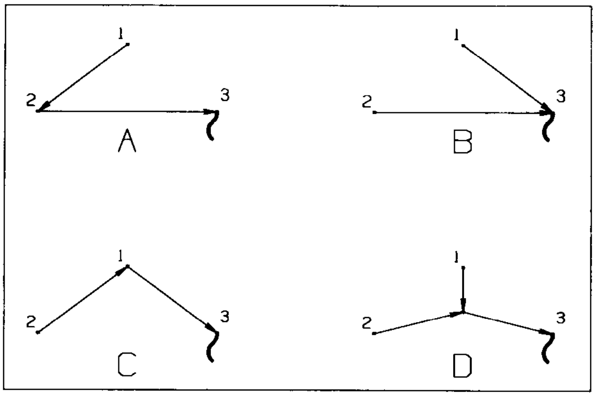

Figure 1. Network configurations.

A more complete geometric description of the fourth case is provided by the sketch of Figure 2. The variables shown in the sketch are listed in Table 2 and their mathematical descriptions are to be found in Table 1. The asterisk superscript on a variable in Table 1 indicates that the associated formula yields an optimal value for the variable. Three analyst-specified control points form the location triangle of Figure 2. If this location triangle has zero area then the control points are collinear and the connecting road solution may be drawn by inspection.

The road segment costs per unit length are also specified by the analyst. Continuing to follow Launhardt's development, these costs may be used to form the sides of a cost triangle as shown in Figure 3. If one of the costs equals or exceeds the sum of the other two then the cost triangle cannot be formed and its formula calculated area (via eq. 8) will be either zero or imaginary. Under these conditions no portion of the road segment exhibiting the larger unit cost will be built. Resulting configurations are shown in the first three cases of Figure 1; e.g., case A occurs when the per unit length cost of building and using road segment 2 equals or exceeds the sum of the two unit length costs associated with segments 1 and 3. These first three configurations are also optimal when the cost and location triangles satisfy a particular mathematical relationship to one another [4]. These relationships are expressed by equations 9. When the right-hand side of one of these formulas yields a value greater than or equal to zero, then the corresponding road segment is assigned zero length.

The complete solution algorithm may be described with reference to Figure 4. The analyst must provide the Cartesian coordinates of the three control points and the cost per unit length for the three road segments. The area of the location triangle is calculated using the coordinate area formula (eq. 3). If the area is zero then the three control points are collinear and the subsequent procedure is not applicable. If the area is non-zero but negative in sign then the control points are re-ordered so that they form a counterclockwise sequence with positive area. (The cost per unit length variables must also be re-indexed if the control points are re-ordered.) If the area of the cost triangle is not a real positive number then the network configuration is determined by eliminating the road segment that has the largest cost per unit length. For a cost triangle with a real positive area a check is performed on the mathematical relationship between the cost and location triangles that would identify one of the first three configurations of Figure 1 as being optimal. If these final checks are negative then the optimal junction point location is interior to the location triangle and is calculated using equations 13 of Table 1.

EXAMPLE IN APPLICATION

Launhardt [6] gives an example in which he applies the Principle of the Node. He solves his example by skillful geometric construction and careful measurement. But no amount of care can entirely eliminate measurement error from this protracted procedure. The input data for this particular problem are listed in Table 3 along with Launhardt's solution. The correct answer to the same number of

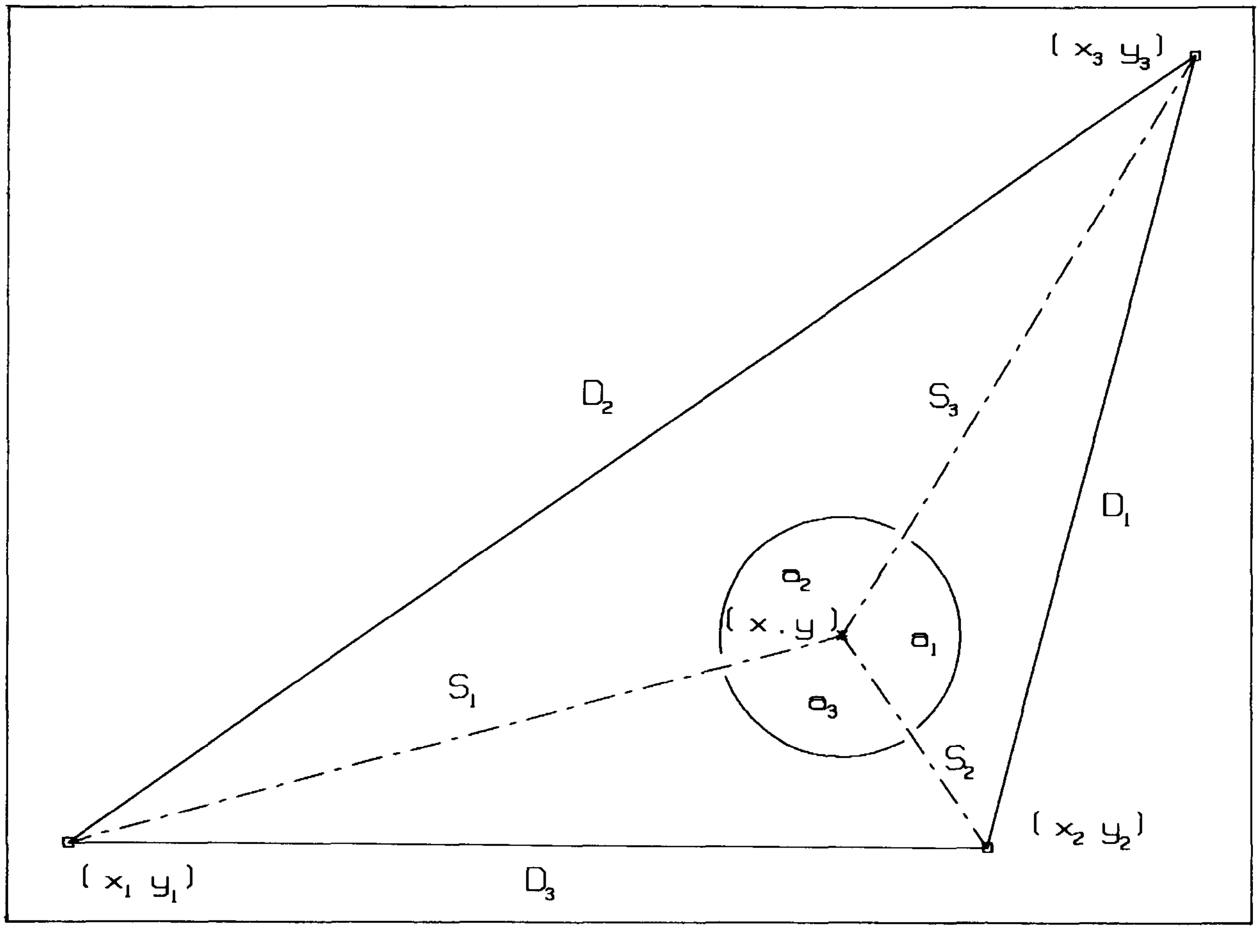

Figure 2. Geometry of the location triangle.

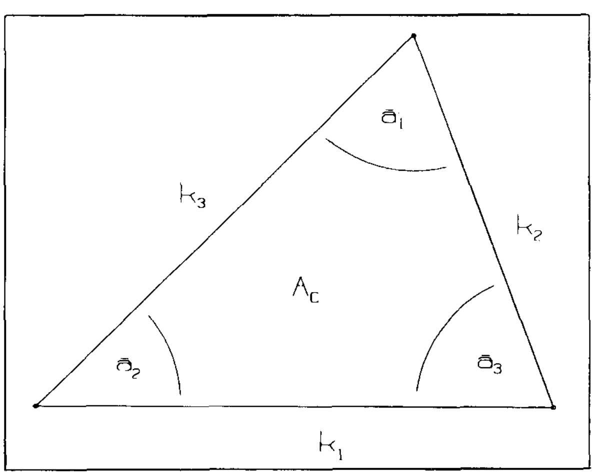

Figure 3. The cost triangle.

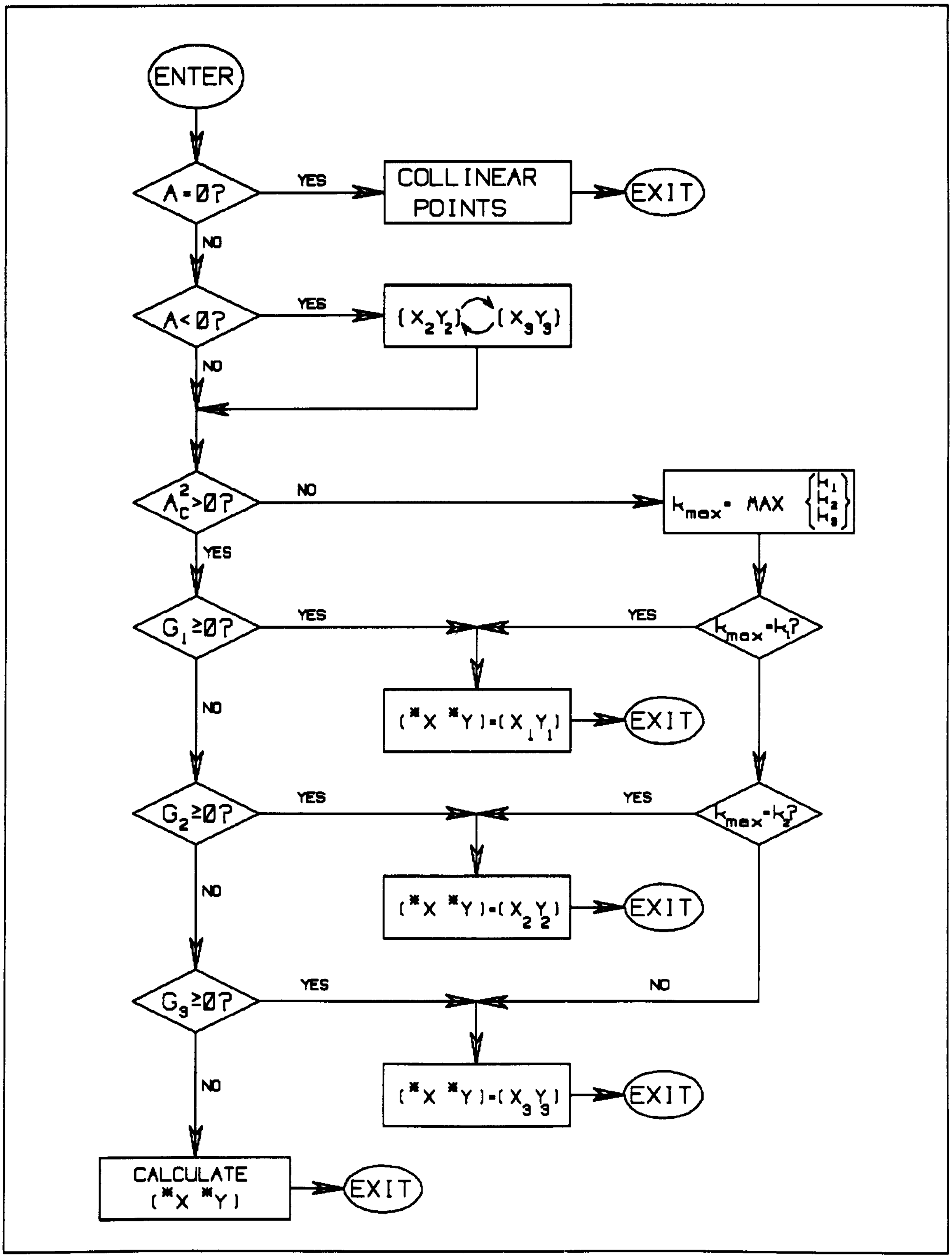

Figure 4. Flow chart of the mathematical algorithm.

PROBLEM DATA FROM LAUNHARDT

DISTANCES			COSTS PER UNIT LENGTH		
D_1	D_2	D_3	k_1	k_2	k_3
9	15	10	12.90	9.95	13.30

AUTHOR ASSIGNED CARTESIAN COORDINATES

(x_1,y_1)	(x_2,y_2)	(x_3,y_3)
(0, 0)	(10, 0)	(12.200, 8.727)

OPTIMAL VALUES FROM THE ALTERNATIVE PROCEDURES

	LAUNHARDT'S GEOMETRIC ANALYSIS	AUTHOR'S MATHEMATICAL ANALYSIS
$^*S_1$	8.67	8.71
$^*S_2$	2.72	2.76
$^*S_3$	7.56	7.50

Table 3. Input data and solutions for the Principle of the Node example for Launhardt [6].

decimal places is given for comparison. This latter answer is obtained using the algorithm and equations presented here. Differences between the two answer sets are entirely due to measurement error associated with Launhardt's procedure. Other solution procedures based on iterative numerical techniques might also be applied [5]. These latter techniques require additional programming skills and typically converge to the optimal solution only in the limit. From among these alternatives, both accuracy and computational efficiency make the mathematical algorithm and closed-form solution of this paper the preferred choice for contemporary forest engineering applications.

CONCLUDING OBSERVATIONS

The mathematical procedure presented in this note should contribute to the continued development of computer-based timber harvesting models. One such model, built to identify the optimum economic access route into a harvest unit, is currently restricted to one centralized landing [2]. The analytic procedure given here will facilitate the evaluation of additional landings. In another model, the environmental damage associated with stand access during harvesting operations was perceived to be a major cost [3]. Reduction of skidtrail and truck road length was identified as an important engineering design objective. In operations of this type the new procedure has the potential to provide rapid identification of shorter timber extraction routes.

An essential element of any application of this procedure is the degree to which the major assumption of flat, uniform terrain is met. It is clear that smaller areas, as in the harvest unit example above, are generally more likely to meet this assumption. The utility of the algorithm under more restrictive conditions, such as those presented by heavily dissected topography or a mosaic of soils of very different engineering properties, may be quite limited. Likewise, what role, if any, it might play in connection with more elaborate and extensive road network design models such as those developed by Nieuwenhuis [8], Douglas and Henderson [1], Shiba and Lîffler [11], Tan [12], and Liu and Sessions [7] remains to be determined.

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NOTE

A copy of the FORTRAN source code may be obtained from the author either via INTERNET at: greulich@u.washington.edu or by posting a formatted high density (1.44 Mbyte) 3 1/2" diskette together with a self-addressed pre-paid mailer to the author. Copies of the proof that equations 13 satisfy the necessary condition for an optimum can also be obtained from the author.