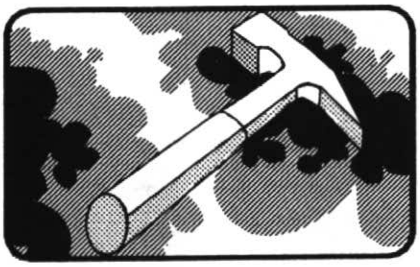
Articles



Non-linear Dynamics and Chaos: Potential Applications in the Earth Sciences

Gerard V. Middleton
Department of Geology
McMaster University
Hamilton, Ontario L8S 4M1

Introduction

The purpose of this short review article is to draw the attention of geologists to a new approach to science that has emerged over the last twenty years. This new way of approaching science can itself be described as a new discipline within science, which is generally given the name "non-linear dynamics". The new discipline cuts across old disciplinary lines, and is in a period of explosive (exponential) growth (Figure 1). Though the first papers in the field are generally considered to be those published in the 1960s (Lorenz, 1963; Hénon and Heiles, 1964; Smale, 1967), and the field did not become popular until the 1970s (Ruelle and Takens, 1971; Yorke and Li, 1975; Hénon, 1976; May, 1976; Swinney and Gollub, 1978), the number of papers published now totals in the thousands, and several systematic expositions, symposia, and reprint volumes on the subject have been published (Bergé *et al.*, 1986; Schuster, 1984, 1988; Thompson and Stewart, 1986; Devaney, 1986: see other works listed in the bibliography). Pioneer works in the field are cited tens of times per year: for example, the paper by Hénon and Heiles (1964) was recently featured as a "citation classic" in *Current Contents* and has been cited over 500 times. Series of review articles have appeared in *Science* (Pool, 1989) and in *The New Scientist* (Percival, 1989). There has even been an excellent popular book-length account (Gleick, 1987), parts of which were originally published in *The New Yorker*. The topic, which originated in mathematics and physics, has been quickly shown to have significant applications in engineering, chemistry and biology, but the number of articles published about geological applications is still very small. There are indications that this situation is about to change, that there are many possible applications of non-linear dynamics in the earth sciences, as in the other sciences, and that descriptions of them are about to appear in geological journals.

The term "non-linear dynamics" is used to describe any system that can be described by a non-linear differential equation. Almost all the theoretical work so far has been on non-linear ordinary differential equations, though chaotic phenomena are known to be exhibited by physical systems (*e.g.*, convecting fluids) which are governed by non-linear partial differential equations. It is more difficult to define exactly what constitutes chaos: many authors consider that a chaotic system is one which is fully deterministic, but which evolves with time in a way that is very sensitive to the specified initial conditions. As the initial conditions of natural systems can never be specified exactly, this means that, for practical purposes, the future behaviour of the system cannot be predicted accurately for far into the future.

It is convenient to introduce the subject by considering first some of the simplest possible examples of differential and difference equations.

Differential and Difference Equations

A very simple differential equation, and one that is certainly familiar to geologists, is that which describes exponential growth or decay:

$$dx/dt = r\,x \qquad (1)$$

The equation states that the rate of growth (or decay) of the variable x is proportional to the value of x itself, and that r is the coefficient of proportionality. Growth takes place if

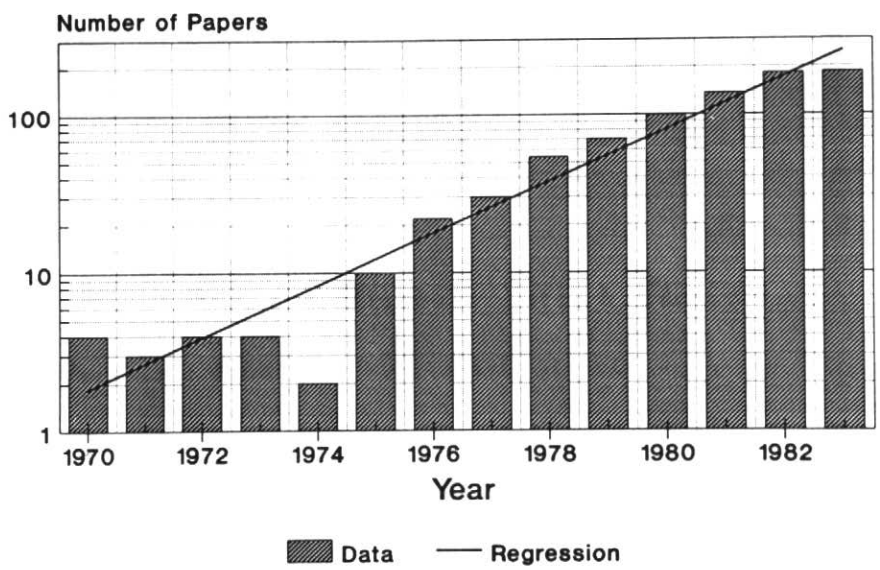


Figure 1 *Exponential growth in number of publications on chaos. Hao (1984) gives a bibliography of 841 references, from which these data were obtained. They show that the number of papers published on chaos has doubled almost every two years since the mid 1970s. If this rate has continued to the present, there are now about 1000 papers on chaos appearing each year.*

r is greater than zero, and decay if r is less than zero. Note that the equation is linear, since the rate of change of x depends only on x, not on some power of x. The solution (or integral) of this equation is well known:

$$x = x_o e^{rt} \qquad (2)$$

In this case, it is easy to integrate equation (1), and obtain the "analytical" solution given by equation (2). Many differential equations are not so easily integrated, however, and in these cases an approximate solution can be obtained by considering the corresponding *difference equation*, which gives the value of x after (n + 1) increments of time, in terms of its value after n increments. For equation (2) this would be written:

$$x_{n+1} = x_n + r\,x_n \qquad (3)$$

or

$$x_{n+1} = R\,x_n$$

where R = r + 1. In these equations, x_n means the value of x at t_n, and might also be written $x(t_n)$. Difference equations need not necessarily be considered only as approximations to differential equations. They may be used as mathematical models for phenomena which are naturally divided into a series of equal, discrete steps. For example, Equation (3) may be considered to represent the growth of a biological population, in which reproduction takes place once a year, and the size of the population next year is proportional to its size this year. For negative r, the value of x decays with time towards zero. For positive r, x increases exponentially without limit — a situation that is inherently unlikely in most practical applications. A more generally applicable equation is developed in the next section.

The Logistic Equation

Though Figure 1 demonstrates that the number of papers appearing on chaos was for a while doubling every two years, we know that this cannot continue far into the future. Instead we expect that the number appearing each year will begin to level off, perhaps approaching some limiting value x_{max}. If we use this value to scale x (so that the range of values of the scaled x is from 0 to 1), we can write a differential equation describing this type of limited growth (or decay):

$$dx/dt = rx(1\text{-}x) \qquad (4)$$

This is the well-known *logistic equation*, which has been applied in economics, and particularly in biology. May (1976) applied it to population dynamics and was one of the first to emphasize its extraordinary character. It is often seen in its alternative forms:

$$dx/dt = 4bx(1\text{-}x) \qquad (4a)$$

or

$$dy/dt = 1 - \mu y^2 \qquad (4b)$$

These equations are non-linear — in fact, they are just about the simplest example that can be given of a non-linear differential equation. Many non-linear differential equations have no known analytical solution, but the logistic equation is an exception. Equation (4) has the solution:

$$x(t) = \frac{x_o}{x_o + (1\text{-}x_o)e^{-rt}} \qquad (5)$$

This is shown graphically in Figure 2. Note that it is a family of regular, smooth curves, and that future values of x are completely predictable, if the initial value x_o and the growth coefficient r are known.

In just the same way that we wrote a difference equation corresponding to exponential growth or decay, we can also write one corresponding to logistic growth or decay. The equation is:

$$x_{n+1} = r\,x_n(1\text{-}x_n) \qquad (6)$$

It might be thought that this equation would yield values closely approximating those given by Equation (5), but in fact it does not. Instead it shows an astonishingly complex range of sequences of x, for some values of r.

Equation (6) is variously known to mathematicians as the difference form of the logistic equation, the "logistic map", or even the "iterated map on the interval" (Collet and Eckmann, 1980). Its properties have been fully explored only in recent years. It has become one of the paradigms of the new science of non-linear dynamics. Though computed sequences of x converge rapidly on one or more values for some values of r, for other values of r the computed values of x display an apparently random variation. This can be clearly seen by plotting computed values of x against r, where equation (6) is iterated many times for each value of r, and points are plotted only after a number of iterations, to allow convergence to take place (Figure 3).

Among the fascinating aspects of Figure 3 are the following:

(i) parts of the map show regular, and other parts highly irregular, behaviour. A sequence of x values generated in one of the "chaotic" regions of the map would appear to an observer to be a random sequence of x values — even if the observer applied some of the well-known

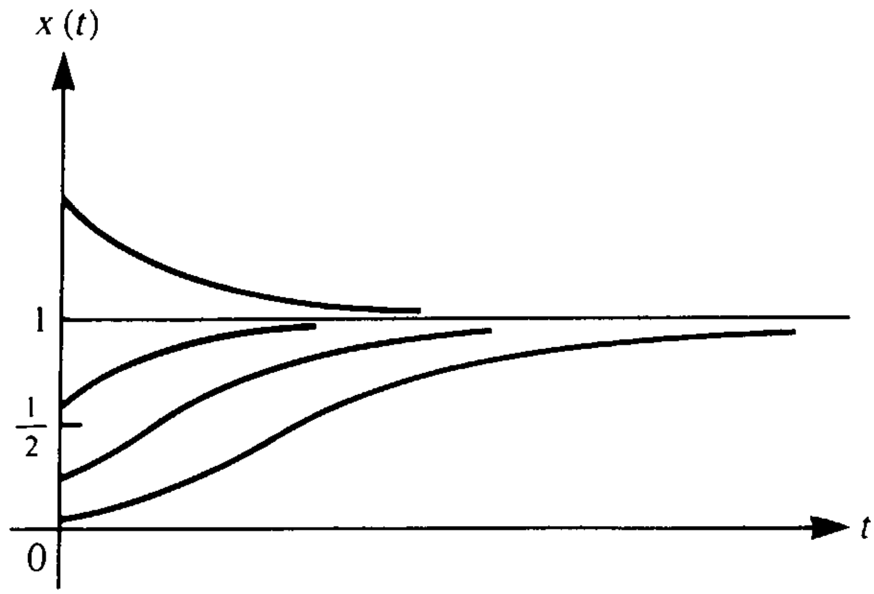


Figure 2 *Graph of the Logistic equation. The different curves correspond to different choices for the parameter r. No matter what the starting value of s and r, x always converges on the value x = 1. For x_o larger than 1, the curves show exponential decay, for x_o close to zero, the curves show an early period of exponential growth, then flatten out and approach the limit of 1 asymptotically.*

statistical tests for "randomness" of a time series. Yet the sequence is strictly determined by the initial values chosen, and so it is a fully deterministic and predictable sequence.

(ii) though it appears from Figure 3 that chaotic behaviour is observed for values of r between about 3.57 and 4, chaotic behaviour is exhibited only for certain values of r in that range. Close examination of the chaotic "region" of the map, shows that there are very many values of r in the range of 3.57 to 4.0 which show non-chaotic behaviour. Enlargement of parts of the map reveals features very similar to those of the complete map — in other words, the map shows aspects of self-similarity strongly reminiscent of fractals (Figure 4). In fact, it can be shown that the logistic map *is* fractal in the chaotic region.

(iii) transition from regular to chaotic behaviour is preceded by a series of "bifurcations" — that is, by a change (as r is increased) from convergence on a single value of x (for a given r) to convergence on two alternating values of x, to four alternating values of x, and so on. When the logistic map is plotted with a geometric scale for r, it can be seen that the bifurcation points occur at nearly regular, geometrically decreasing values of r.

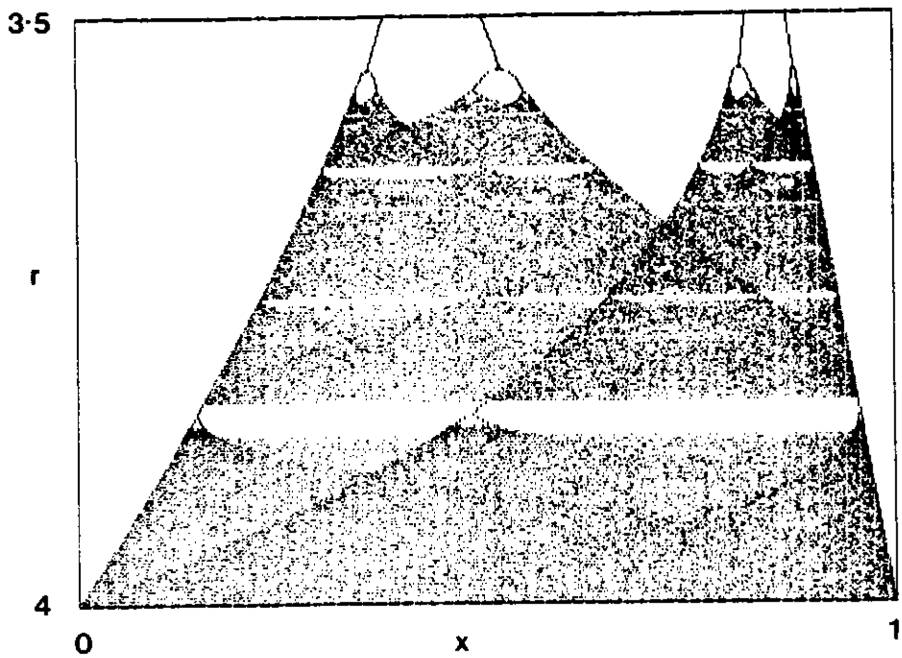


Figure 3 *The Logistic Map. The map is shown for the range x = 0 to 1.0, and r = 3.5 to 4.0. The first bifurcation takes place at r = 3.0, and the second at r = 3.45. The chaotic region begin at r = 3.57.*

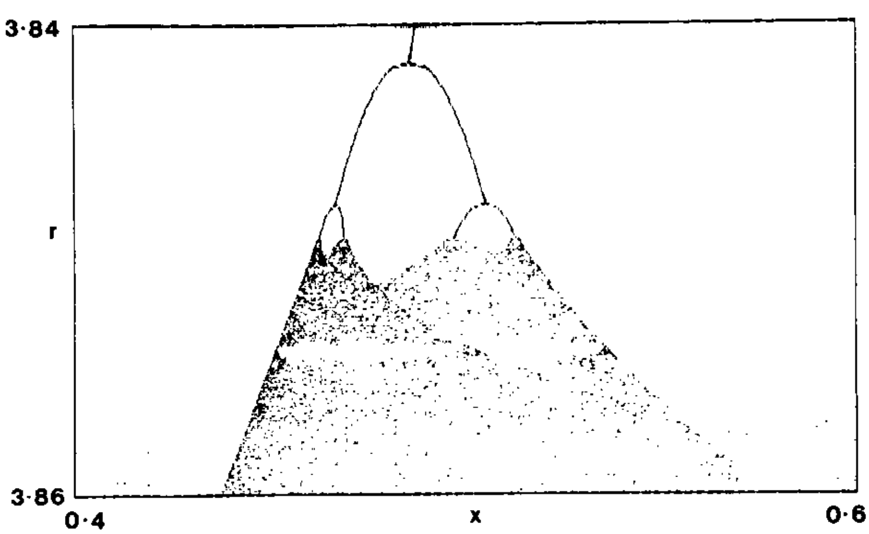


Figure 4 *A small part of the logistic map, enlarged. The region shown is from x = 0.4 to x = 0.6, and from r = 3.84 to r = 3.86. Figure 2 and 3 were produced using a screen dump on a rather primitive personal computer (the TI Pro). Higher resolution plots can be found in Collet and Eckman (1980).*

It is beyond the scope of this article to explore the logistic map further. The complex behaviour exhibited by this very simple difference equation was the subject of early investigations by May (1976), Feigenbaum (1980) and others, and has been given systematic book-length exposition by Collet and Eckmann (1980) and Devaney (1986). The investigations raised important questions about the stability of differential equations and the dynamic systems that they represent, and about the definition of "chaotic" as opposed to "regular" behaviour of these systems.

In this case, chaotic behaviour is shown only by the difference form of the logistic equation, not by the differential form. Chaotic solutions cannot be avoided, however, by choosing a different finite difference approximation: other common approximations show similar (though not identical) maps (Prüfer, 1985). It is not true, however, that only finite difference equations show chaos. Many sets of first-order ordinary differential equations involving at least three dynamic variables (or higher-order differential equations that can be reduced to this form) are known to yield chaotic solutions. The study of such systems is now generally called "non-linear dynamics". Though the laws of classical dynamics naturally give rise to chaotic systems, the term "non-linear dynamics" is now used to apply generally to the study of systems of non-linear differential equations, and is not restricted to mechanical systems.

Stability of Dynamic Systems

Elementary texts on the theory of ordinary differential equations generally include some discussion of the question of stability. The simplest system, discussed in both freshman physics and calculus courses, is the classical oscillator or pendulum. The equation of motion for the simple pendulum is

$$d^2\Theta/dt^2 = -(g/l) \sin \Theta \qquad (7)$$

where l is the length of the pendulum, g is the acceleration due to gravity, and Θ is the angle that the pendulum makes to the vertical. If we remember that for small angles (measured in radians) $\sin \Theta \approx \Theta$, we can state simply that the angular acceleration of the pendulum is proportional to the angle of deflection from the vertical. Equation (7) then becomes a (second-order) linear equation.

It is useful to express second-order equations, such as those that generally arise in dynamics (because of the emphasis on accelerations), as an equivalent system of two first-order equations. We do this by substituting $d\Theta/dt = y$, which leads to the system:

$$d\Theta/dt = y \qquad (8)$$
$$dy/dt = -(g/l) \sin \Theta$$

The motion of the pendulum can then be conveniently represented by plotting the angular velocity ($d\Theta/dt$) against the angle of deflection (Θ). The space defined by these two co-ordinates is called a state space or *phase space*, and discussions of phase spaces (of two or more dimensions) loom large in most papers on non-linear dynamics. In such a space, the locus of points describing the motion resulting from some initial condition (*i.e.*, some initial angular deflection and velocity) is a "trajectory" in the phase space. Though the locus changes with time (as both angular velocity and angle of deflection change with time), time is not represented explicitly on the graph. For the example given, the trajectory in phase space will be an ellipse (almost circular for small deflections), whose dimensions are fully determined by the coefficient (g/l) and the initial conditions.

The simple pendulum is, by definition, frictionless, so there is no progressive change in the trajectory with time. If we introduce friction, however, the trajectories are no longer ellipses, but become spirals approaching the origin (no deflection, no velocity) at large times (Figure 5a). Such a point is called an *attractor* because the trajectories seem to be attracted to it. Attractors are characteristic of dissipative (non-conservative) systems. They need not be points, as can be seen from the example of a pendulum which is periodically "kicked" (the motion of a child in a swing that is periodically pushed is a similar phenomenon). So long as the periodic pushing continues, the swing (or pendulum) never comes to rest, but after an initial period of adjustment it usually settles into a stable trajectory, which will be represented by a particular closed loop in phase space. If the initial condition is characterised by smaller (or larger) deflections and velocities than those found on this loop, the deflections and velocities gradually increase (or decrease) until the stable condition is achieved. This corresponds to trajectories which spiral outward or inward towards the stable loop, which is called a *limit cycle* (Figure 5b, 5c). Limit cycles are just another kind of attractor. We can designate point attractors as having dimension zero, line attractors (loops) as having dimension one, *etc.* Many dynamic systems display attractors which are tori (doughnut-shaped) surfaces — therefore of dimension two (Figure 5d).

Notice that if we plot a series of points in phase space, measured at closely spaced times, they will first approach the attractor and then fall along the attractor (strictly speaking, they approach the attractor asymptotically). Eventually they may begin to repeat themselves. If the particular phase space of interest were three dimensional (as it would have to be to produce a two-dimensional attractor) we could obtain a simplified picture of what was happening by considering the intersections of the trajectory with a single well-chosen plane. For example, if the attractor is a torus and the plane intersects it in a small circle, the trajectory will intersect the plane at a point on the circle each time it returns. Such planes of intersection are called *Poincaré sections* of the phase space.

Scientists interested in sets of differential equations have long taken an interest in the geometry of phase spaces, because it is possible to represent conditions for system stability or instability in such spaces. For two-dimensional linear systems the conditions are well known (*e.g.*, Boyce and DePrima, 1986, chapter 9).

A system of such equations can be written

$$\begin{aligned} dx_1/dt &= ax_1 + bx_2 \\ dx_2/dt &= cx_1 + dx_2 \end{aligned} \tag{9}$$

or in matrix notation

$$dx/dt = \mathbf{A}\,x$$

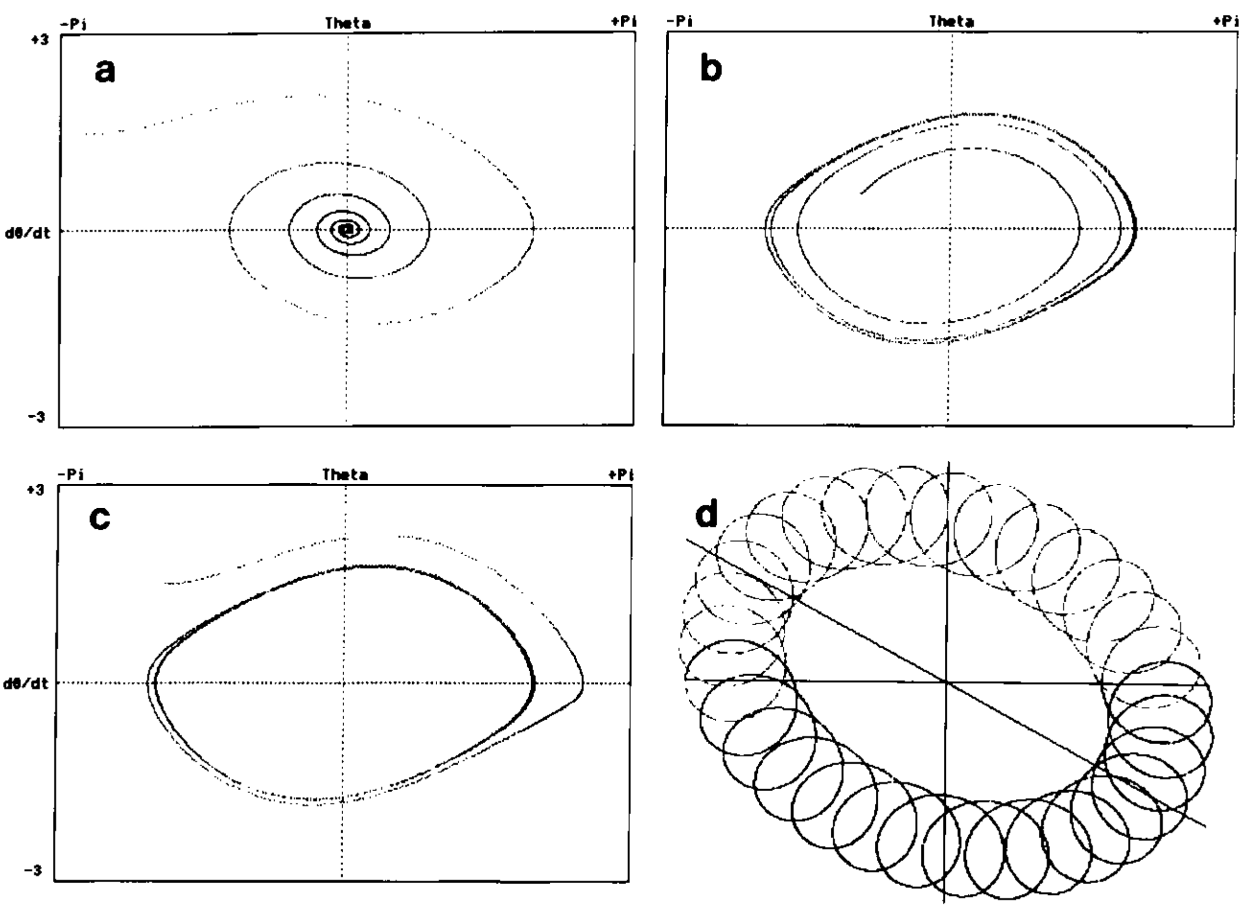


Figure 5 *Attractors shown by oscillators. The point attractor* (**a**) *is shown by a simple, damped oscillator. The limit cycle* (**b** *and* **c**) *is shown by a forced oscillator. The same limit cycle is approached asymptotically both from inside (b) and from outside* (c) *the cycle. The torus* (**d**) *corresponds to a limit cycle in a three dimensional phase space.*

The solution is obtained from the characteristic equation

$$\det |\mathbf{A} - p\mathbf{I}| = 0 \quad (10)$$

where **I** is the unit matrix. This determinant is simply a polynomial in p: the values of p are called the characteristic roots or eigenvalues of the system. In the case of two variables, the equation is a quadratic

$$p^2 - (a+d)p + (ad-bc) = 0 \quad (11)$$

with two roots given by the familiar formula

$$p = \{(a+d) \pm\sqrt{((a+d)^2 - 4(ad-bc))}\}/2 \quad (12)$$

We notice that both of these roots might be complex, *i.e.*, involve the square root of a negative number.

When we speak of the *stability* of a system, we are generally concerned with the stability in the neighbourhood of equilibrium points (or steady states) of the system. These are points for which the derivatives are all equal to zero. In phase space trajectories will either converge toward such a point (this will be a stable equilibrium point) or diverge from the point (an unstable equilibrium point), or there will be some trajectories converging on the point and some diverging from it (also an unstable equilibrium point). The stability of an equilibrium point, $\mathbf{x}^*$, is determined by considering the way that $\mathbf{x}$ varies with time in the immediate vicinity of $\mathbf{x}^*$, that is by the equations:

$$d\mathbf{x}/dt = \mathbf{A}(\mathbf{x} - \mathbf{x}^*) \quad (13)$$

where **A** is now a matrix of partial derivatives, evaluated at $\mathbf{x}^*$ (at the point itself, $d\mathbf{x}^*/dt = 0$). Such a matrix is called a *Jacobian*. The set of equations (13) is mathematically analogous to the set (9), and the solution is again obtained by computing the eigenvalues. The eigenvalues can be used to classify the type of equilibrium point: (i) if the eigenvalues are complex conjugates (with both real and imaginary parts), then the point is a *focus* and trajectories in phase space spiral in or out from it; (ii) if both eigenvalues are real and have the same sign, the point is a *node* and trajectories converge on or diverge from it; and (iii) if both eigenvalues are real but have opposite signs, the point is a *saddle point*. Foci and nodes may be either stable or unstable, depending on whether trajectories converge on or diverge from the point, but saddle points are always unstable.

Though this analysis assumes linearity, it may be used to examine the stability of equilibrium points in non-linear systems, since it is nearly always possible to make a linear approximation to the non-linear equations in the immediate vicinity of the equilibrium points.

Bifurcations and Chaos

Linear sets of differential equations can always be solved analytically, but this is not always the case for sets of non-linear equations. Generally, however, it is possible to determine a great deal about the behaviour of non-linear systems from studying the equilibrium points and their stability. Non-linear equations may also be solved (approximately) using numerical techniques.

The number and stability of equilibrium points of a set of differential equations can be observed to change as the coefficients of the equation change. Such changes represent fundamental, and sometimes abrupt, changes in the behaviour of the system: they are called *bifurcations*. The type of change represented by a bifurcation may be classified by examining the eigenvalues of the relevant Jacobian before and after the bifurcation. Many bifurcations have been named (*e.g.*, Hopf bifurcation, pitchfork bifurcation) and a great deal of study has been devoted to them. A concise summary is given by Bergé *et al.* (1986, Appendix A).

One possible result of certain types of bifurcations is replacement of a single stable equilibrium point by two such points. Then, for example, a dynamic system oscillating with a stable period, T, might be replaced by one alternating between two modes of oscillation, so that the period of a complete cycle changes from T to 2T. It is found that this is a feature common to several different systems, and that the *period doubling* may be repeated several times as the controlling parameter is changed. The magnitude of the parameter change needed to produce a period doubling decreases in a way that approaches a regular geometric series. This was first discovered for the logistic map (it is illustrated in a crude way by Figure 4). The ratio between successive changes is found to approach a limit which is a fundamental constant, characteristic of not only of the logistic map, but of many different systems *(Feigenbaum's constant).*

After a series of such period doubling, a further change of the control parameter often causes the system to begin to display a wildly erratic type of behaviour, which is generally called *chaos*.

Chaos is not displayed by all systems which display instability. For example, the simple damped pendulum has an unstable equilibrium point, when the pendulum points straight up (with the bob on a rigid rod vertically *above* the support). If displaced slightly from this point, the pendulum swings down and begins to oscillate above its other equilibrium point (with the bob vertically *below* the support) and eventually comes to rest at this point. Instability in this case gives rise to perfectly regular, predictable oscillations about a stable equilibrium point. The pendulum does not normally display chaos even when it is "periodically kicked": it settles down into the regular orbit characterized by the limit cycle in phase space. But a pendulum that is "pumped" (for example, by vertically oscillating the support) can display amazingly complex motions, which have only recently been investigated (Greene, 1986).

What is particularly interesting about systems that do display chaos is that extremely complex behaviour may be shown by relatively simple systems (systems of only three differential equations, for example). The behaviour of these systems, though very complex, is perfectly predictable, in the sense that it is easily reproducible from one numerical experiment to the next. But a time series of measurements on such a system, made by an observer who did not know the controlling equations, would appear to be varying in a random and unpredictable manner.

This does not mean that such systems are necessarily completely unstable. Dissipative systems that show chaotic behaviour show convergence on attractors, just as non-chaotic dissipative systems do. But the attractors are not simple stable points, limit cycles, or surfaces. Instead, they have very complex geometries, and are described as *strange attractors*. Trajectories in phase space converge toward such attractors, but two trajectories starting from two points initially very close together diverge rapidly (in fact, at an exponential rate, which may be measured by a coefficient called the *Lyapunov exponent*). Since the attractor has only a finite size in the phase space, however, the trajectories may eventually come close to each other again.

How is it possible for trajectories to diverge exponentially, yet remain on an attractor which occupies only a part of phase space? The answer is given by the "horseshoe" model first proposed by Smale (1967): diverging trajectories are stretched out in space (like a straight piece of hot iron), then folded back on themselves, and the process is repeated, essentially for ever. As a result, strange attractors display very complex geometry: many can be demonstrated to show fractal properties (self similarity at different scales), and their fractal dimension can be deduced mathematically, or determined empirically from time series generated by the system. It should be emphasized that computer generated pictures of strange attractors (such as Figure 6) show only a few trajectories lying close to the attractor. The complex nature of the surface is shown by enlarging a part of the attractor to larger scales, and computing a much larger number of trajectories.

The classic example of a system with a strange attractor is that first described by Lorenz (1963). Lorenz, a meteorologist working at the Massachusetts Institute of Technology, was led to this system by seeking the simplest representation possible of the very much more complex system of partial differential equations which represents motion in a thermally convecting atmosphere. Subsequent workers have suggested two physical systems which correspond more directly to the Lorenz equations:

(i) a water wheel with leaky buckets (Lorenz, 1979; Gleick, 1987, p. 27); and

(ii) thermal convection of a viscous fluid under gravity in an upright closed circular tube (Tritton, 1988, chapter 17).

The Lorenz system (in non-dimensional form) is:

$$\begin{aligned} dx/dt &= \sigma\,(y-x) \\ dy/dt &= rx - y - xz \\ dz/dt &= xy - bz \end{aligned} \qquad (14)$$

The equations appear quite simple, though they contain the non-linear terms xz and xy (σ and r are numerical coefficients). They cannot be solved analytically, and therefore Lorenz began to study their behaviour using a numerical approximation method on a primitive electronic computer. The story of how this led him to discover their chaotic behaviour, and the "first" strange attractor is well told in Gleick's book (1987, p. 11-31).

The behaviour of this dynamic system depends upon the control parameter r. As r is increased above unity, the equilibrium points undergo a series of bifurcations. When $r = 28$, the behaviour of any of the variables x, y, or z is chaotic. Lorenz discovered by accident that if he used two initial conditions (x,y,z) which differed only in the fourth significant figure, the numerical solutions, though similar for short times (small numbers of numerical iterations) rapidly diverged and became totally dissimilar at large times. This "sensitivity to initial conditions" has subsequently become accepted as the most important defining characteristic of chaotic systems. It has, of course, immediate relevance to problems such as long range weather-forecasting. If the governing equations of weather systems are indeed systems similar to the Lorenz equations (though more complex), then it will never be possible to predict the weather accurately far into the future, because the initial conditions can only be known to a limited degree of approximation.

The Lorenz attractor is shown in Figure 6: the Lorenz system has become the paradigm of a simple three-dimensional system with a strange attractor (rivalled in popularity only by the system described by Rössler, 1976) and has been given monographic treatment by Sparrow (1982). For a short article by Sparrow on this system, and for illustration of many other attractors, see the book edited by Holden (1986).

Why Is Non-linear Dynamics Popular?

The discussion given above may strike some readers as rather abstract: why is it that such seemingly obscure properties of differential equations have suddenly become a major field of investigation in the physical sciences?

One answer is that almost all of the physical sciences abound with examples of complex, irregular patterns of behaviour. Perhaps the most commonplace example is that of turbulence in fluid flows. Yet science has previously either ignored these phenomena, or treated them as being "stochastic", that is, as arising from a large number of unknown (perhaps even unknowable) causes, whose net effect can be described only by the addition, to the otherwise deterministic variables, of a non-deterministic "random variable". The findings of non-linear dynamics came as a shock to many classically trained scientists, first because most of us had not been at all aware of the strange behaviour that can be exhibited by sets of non-linear equations (yet we know that nature is full of non-linearity), and second, because we did not realize that extremely complex, apparently "random" behaviour might arise from very simple sets of deterministic governing equations (Ford, 1983).

For most scientists, this shock is at once dismaying and exhilarating. It is dismaying because it suggests even closer limits on the power of classical methods than we had previously realized; and it is exhilarating because it suggests theoretical and experimental approaches to phenomena which had previously defied rational investigation. The best example of a new line of investigation spawned by non-linear dynamics is probably the studies that have been (and are being) carried out in the generation of fluid turbulence. It has been shown that the way that instabilities arise and multiply in some fluid systems (Taylor vortices in the annulus between rotating cylinders, and vortices arising from Rayleigh-Bénard convection) do in fact display exactly those properties (bifurcation, period doubling, *etc.*) that might be expected if the early stages of turbulence are generated by relatively simple non-linear dynamic equations (for a review see Swinney and Gollub, 1978; or the book by Bergé *et al.*, 1986).

The present phase of investigations by physical scientists (as opposed to investigations by mathematicians and theoreticians) seems to be dominated by the attempt to discover examples of chaotic systems in natural phenomena. There are basically two ways to do this: one can either start with the mathematical model, and show that it accurately describes the behaviour of some physical system of interest (*e.g.*, the Lorenz equations have been found to describe thermal convection in an upright circular tube) or one can start with the phenomenon, and show that it displays features typical of non-linear dynamics, rather than those expected from a purely stochastic process. It is also true that it is very difficult to make generalizations about different systems of non-linear differential equations. Such systems must

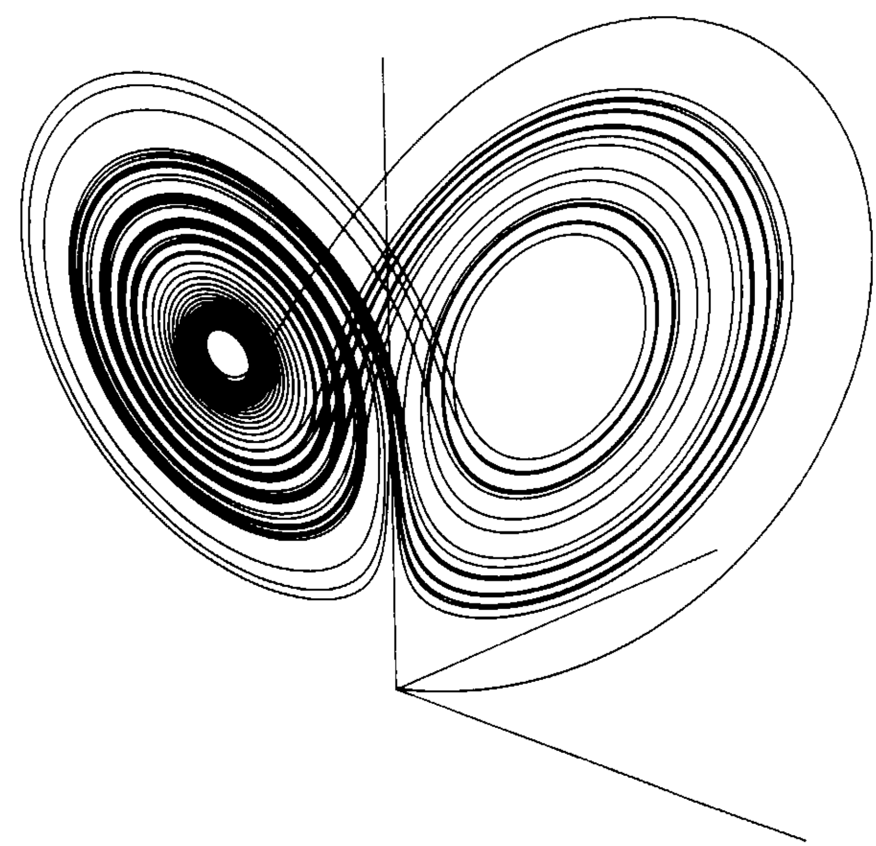


Figure 6 *The Lorenz strange attractor. (From Moon, 1987).*

generally be studied using a combination of mathematics tools and numerical experimentation. A few such systems (the logistic map, the Lorenz equations) have now been investigated in detail, and such studies provide paradigms for the investigation of an essentially limitless number of other systems. The necessary numerical experimentation involved was hardly possible before the development of the digital computer, but it is now well within the scope of even simple personal computers (in fact, the ready access to "real time" graphic display favours the personal over mainframe computers). Chaos is therefore an intellectually exciting new field, and one that is readily open to those whose budgets and intellectual powers are both not unlimited.

Finally, one must add a further, purely aesthetic incentive. The images of non-linear dynamics, so readily generated on a personal computer (Koçak, 1989), have a strange visual enchantment, an enchantment that is felt even by those who understand nothing of the underlying theory. Non-linear dynamics shares this with the science of fractals, which is also a very popular topic, for much the same reasons. There are, in fact, close theoretical ties between the two topics, since strange attractors are typically fractal (Devaney and Keen, 1989).

Possible Applications in the Earth Sciences

The classic investigations of Lorenz on thermal convection in the atmosphere may themselves be considered to have been within the broader field of the earth sciences. Thermal convection within the mantle may also show chaotic behaviour (Stewart and Turcotte, 1989). It is generally believed that magnetic reversals (which show an apparently random time sequence) are ultimately related to convection in the core of the earth, so it is not surprising that the concepts of non-linear dynamics were early applied to this phenomenon. The simplified disk dynamo model (Chillingworth and Holmes, 1980; Tritton, 1989) gives rise to a set of three equations which are almost identical with the Lorenz equations. Sun spots are also a phenomenon related to thermal convection in the outer layers of the sun, and Tai (1987) has found some evidence that the historical record of sun-spot cycles shows evidence for dynamic chaos, as well as regular rhythms.

There have long been questions asked about the long term stability of the solar system (*e.g.*, Moser, 1978). Wisdom (1987) has recently demonstrated chaotic behaviour shown by Hyperion, one of the moons of Saturn, and Sussman and Wisdom (1988) have used a special purpose "supercomputer" (which he calls a "numerical orrery") to predict that the motion of Pluto will become chaotic within 20 million years or so. The orbits of meteorites and other bolides that strike the earth, are also probable examples of wild (chaotic) behaviour exhibited by solar system non-linear dynamics. It has also been argued (Laskar, 1989) that sensitive dependence on initial conditions in solar system dynamics makes it impossible to determine the nature and duration of Milankovic cycles far back in geological time: if true, this greatly weakens the case of those who claim to identify such cycles in Mesozoic or Paleozoic rocks by analogy with Pleistocene cycles (a popular account has been given by York, 1989).

It is known that certain chemical systems are governed by non-linear equations that show complex periodic and chaotic behaviour. The ability of such systems to produce regular patterns, as well as patterns with fractal properties, has been studied for some time, and under the leadership of Prigogine it has become a special subdiscipline of "Self-Organization" studies. Prigogine has himself written technical (Nicolis and Prigogine, 1977), semi-technical (Prigogine, 1980) and non-technical (Prigogine and Stengers, 1984) books on this topic, and there have been three symposia devoted at least in part to geological applications (Nicolis and Baras, 1984; Nicolis and Nicolis, 1987; the last is not yet published, but was reported by Fowler, 1988). See also the collection of papers recently published in the *Journal of Chemical Education* (Soltzberg, 1989).

The possibility that volcanic eruptions and the seismic tremors associated with them are governed by non-linear dynamics has been discussed by Shaw (1988). A model for earthquakes as chaotic phenomena seems to have been first proposed by Ito (1980): recent studies are reported by Scholz (1989).

River meanders are an example of a phenomenon that clearly shows both surprising regularity and complex, irregular behaviour (they are discussed, rather inconclusively, by Gleick, 1987, p. 197-198). Furbish (1988) has pointed out that local rates of meander migration are controlled by non-linear processes, and has suggested that the irregular behaviour of meanders is an example of dynamic chaos.

Sediment ripples, when first formed on a flat bed, have a very regular geometry, with straight crests and a short wave length. As the ripples further adjust to the flow (without changing the discharge), the geometry of the ripples slowly changes and becomes strongly three dimensional. The phenomenon was carefully studied, using spectral analysis of depth profiles measured across the ripples, by Jain and Kennedy (1974). They interpreted the results in terms of a stochastic model of ripple generation, but it now seems likely that the data can be re-interpreted as an example of chaotic phenomena generated by non-linear dynamics.

Identifying Chaos in Observations

One possible conclusion from Lorenz' original studies is that the irregular fluctuations in the weather, or climate, observed at any point on the earth's surface are not strictly random, but are chaotic fluctuations generated by a relatively simple set of non-linear governing equations. This example, and the examples of magnetic reversals, sun spots, river meanders, and ripple profiles cited above all have something in common. They all pose the question: given a time series (which may, in fact, be a series of measurements made along a spatial, rather than a time dimension), is there any way that we can determine whether its apparently random element has in fact been generated by the non-linearity of a small set of governing equations, rather than by a "truly" random process? If the former is the case, there may be a low-dimensional attractor associated with the system, and it should be possible to estimate its dimension.

Several different methods for the analysis of time series have been proposed as possible answers to this question. It is these methods which seem to have the most immediate application in the earth sciences. The most commonly used method was proposed by Grassberger and Procaccia (1983). It is described by Grassberger in Holden (1986) and by Bergé *et al.* (1986, chapter 6). The basic idea is as follows.

The original time series consists of measurements x_i of some physical variable, separated by equal intervals of time (or space) δt, and numbered $i = 1, 2, 3 \ldots N$. This series can be displayed in a space consisting of the variables $x(t)$, $x(t+\delta t)$, $x(t+2\delta t)\ldots$ up to an "embedding dimension" of m.

For a given choice of m, we now define a "spatial correlation" C(r):

$$C(r) = \lim_{N \to \infty} \frac{n_r}{N(N-1)/2}$$

where n_r is the number of pairs of vectors $(x_i \ldots x_{i+m-1})$ and $(x_j \ldots x_{j+m-1})$ whose distance apart (in the m dimensional space) is less than r. From a sufficiently long time series, C(r) can be determined and and log C(r) plotted against log r. On the assumption that the system has a low-dimensional attractor of (fractal) dimension D, C(r) is expected to vary as r^D at sufficiently small r. For a time series consisting of "white noise", it is found that the slope of the log C(r) - log r plot depends upon the "embedding dimension", increasing in step with m as m increases. For a time series consisting of points which lie on the surface of a strange attractor, however, the slope increases only up to the dimension, D, of the strange attractor. This dimension can therefore be determined by plotting the slope against m.

The method has theoretical justification, and has been found to work quite well for long time series (tens of thousands of data points) generated by known non-linear systems. The computations involved are rather simple, given a computer and a suitable time series. The interpretation is not always so simple, as can be seen by examining the following published examples. Nicolis and Nicolis (1984) used the method to examine a climatic time series based on isotopic measurements made on a deep sea core of Holocene/Pleistocene sediments. They concluded that the record showed evidence of a climatic attractor. Grassberger (1986) criticized their methodology, and advanced an analysis of data indicating no sign of a climatic attractor. Essex *et al.* (1987) analysed a time series of meteorological observations covering a much shorter period of time, but consisting of about 100,000 measurements. They concluded that this record did show evidence for a strange attractor. For a review of the techniques see Henderson and Wells (1988).

Conclusion

Suppose that we do succeed in showing, either from a known system of governing equations, or from an analysis of a time series of observations, that the behaviour of a natural system is consistent with that predicted by non-linear dynamics — are we any further ahead?

Even if the governing equations have been identified, we are still unable to use the equations to make predictions far into the future, for those parts of phase space which display chaotic phenomena. Perhaps it is the case that, even with full understanding of their mechanics, we will never be able to predict the next major earthquake, or the next magnetic field reversal. If the governing equations are known, however, there now exists a variety of established techniques which permit us to understand the system as well as is humanly possible, and certainly better than we could understand it by resorting only to linear approximations.

If all that we have achieved is to identify the system as chaotic, with a strange attractor of dimension D — then our sum total of knowledge is still rather meagre. This information can be used as a guide to further studies, and at least we now know that there is some possibility of achieving further understanding. We know at least that the system is not entirely the product of completely random, and therefore completely unknowable, processes. The dimension of the attractor further suggests the minimum number of dynamic variables required to model the system.

Scientists are used to believing that complete understanding of a physical system implies complete control and predictive ability. We now know that that is generally not the case. That in itself is worth knowing.

Acknowledgments

I thank Bosco Loncarevic and Anthony Fowler for their comments on manuscript versions of this paper, and David Goodings for a particularly thorough review, and for many interesting discussions about chaos during the past year.

Note

GAC Council has approved a Short Course "Non-linear Dynamics, Chaos and Fractals, with Applications to Geological Systems", to be offered at the 1991 Annual Meeting in Toronto. Further information can be obtained from the author of this article.

Bibliography

Non-technical Review Articles and Books

Crutchfield, J.P., Farmer, J.D., Packard, N.H. and Shaw, R.S., 1986, Chaos: Scientific American, v. 255, no. 6 (December 1986), p. 46-57.

Dewdney, A.K., 1987, Computer Recreations. Probing the strange attraction of chaos: Scientific American, v. 257, no. 1 (July 1987), p. 108-111.
Also several other articles in this series.

Gleick, J., 1987, Chaos: Viking Penguin Inc., New York, 352 p.

Jensen, R.V., 1987, Classical chaos: American Scientist, v. 75, p. 168-181.

Percival, I., 1989, Chaos: a science for the real world: New Scientist, 21 October, p. 42-47
Also see other articles in succeeding issues.

Pool, R., 1989, Is it chaos, or is it just noise?: Science, v. 243, 6 January, p. 25-28.
Also see articles in 20 January, 3 February, 17 February, 10 March, and 7 July issues.

Prigogine, I., 1980, From Being to Becoming: Time and Complexity in the Physical Sciences: W.H. Freeman and Co., New York, 272 p.
Emphasis on applications to thermodynamics.

Prigogine, I. and Stengers, I., 1984, Order out of Chaos: Man's New Dialogue with Nature: Bantam Books, New York, 349 p.
A less technical and more philosophical version of Prigogine (1980).

Soltzberg, L.J., 1989, ed., Self-Organization in Chemistry: Journal of Chemical Education, v. 66, p. 187-212.

Stewart, I., 1989, Does God Play Dice? The Mathematics of Chaos: Blackwell Scientific Publishers, Oxford, 317 p.

Tritton, D., 1986, Chaos in the swing of a pendulum: New Scientist, 24 July, p. 37-40.

Technical Review Articles

Bak, P., 1986, The Devil's Staircase: Physics Today, December 1986, p. 38-45.

Eckmann, J.P., 1981, Roads to turbulence in dissipative dynamic systems: Reviews of Modern Physics, v. 53, p. 643-654.

Feigenbaum, M.J., 1980, Universal behavior in nonlinear systems: Los Alamos Science, v. 1, p. 4-27.
This is also the closest there is to a popularization of Feigenbaum's discoveries of universal numbers determining to transition to chaos. Reprinted in Cvitanovic (1984).

Ford, J., 1983, How random is a coin toss?: Physics Today, April 1983, p. 40-47.

Kadanov, L.P., 1983, Roads to Chaos: Physics Today, December 1983, p. 46-53.

Ott, E., 1981, Strange attractors and chaotic motions of dynamical systems: Reviews of Modern Physics, v. 53, p. 655-671.

Ruelle, D., 1980, Strange attractors: The Mathematical Intelligencer, v. 2, p. 126-137.
Translation of Les attracteurs étranges: La Recherche, no. 108, Février 1980, p. 132-144. English version reproduced in Cvitanovic, 1984.

Swinney, H.L. and Gollub, J.P., 1978, The transition to turbulence: Physics Today, v. 31, August 1978, p. 41-49.

Introductory-Level Books

Abraham, R. and Shaw, C., 1982, 1985, 1988, Dynamics: the Geometry of Behaviour: Aerial Press, Santa Cruz, California.
Several paperbound volumes of sketches, with minimal text, illustrating key concepts in nonlinear dynamics (unfortunately much overpriced).

Bergé, P., Pomeau, Y. and Vidal, C., 1986, Order within Chaos. Towards a deterministic approach to turbulence: John Wiley and Sons, New York, 329 p.
Translated by L. Tucherman from the 1984 French edition. This is probably the best single introduction available.

Devaney, R.L., 1986, An Introduction to Chaotic Dynamical Systems: The Benjamin/Cummings Publishing Co. Inc., Menlo Park, California, 320 p.
Best introductory text, written from a mathematical, rather than physical perspective.

Devaney, R.L. and Keen, L., 1989, eds., Chaos and Fractals: The Mathematics Behind the Computer Graphics: American Mathematical Society, Proceedings, Symposia Applied Mathematics, v. 39, 148 p.

Edelstein-Keshet, L., 1988, Mathematical Models in Biology: Random House, New York, 586 p.
Good background text for those who need to brush up their knowledge of difference and differential equations. Includes some treatment of chaotic systems.

Glass, L. and Mackey, M.C., 1988, From Clocks to Chaos: The Rhythms of Life: Princeton University Press, 248 p.
Two Canadian authors give a good introduction to chaos and its application to biological rhythms.

Gould, H. and Tobochnik, J., 1988, An Introduction to Computer Simulation Methods: Application to Physical Systems: Addison-Wesley Publishing Co., Reading, Massachusetts, 2 vols, v. 1, 318 p.
Contains a discussion, with computer programs, of some simple chaotic systems.

Hirsch, M. and Smale, S., 1974, Differential Equations, Dynamical Systems, and Linear Algebra: Academic Press, New York, 358 p.
A mathematics text co-authored by Smale, who has made important contributions to chaos theory.

Holden, A.V., 1986, ed., Chaos: Princeton University Press, 324 p.
A paperback collection of articles amounting to a fairly systematic treatment of chaos, with emphasis on biological applications. Articles by Wolf and by Grassberger give a useful description of techniques to measure chaos.

Koçak, H., 1989, Differential and Difference Equations through Computer Experiments: with a supplementary diskette comtaining PHASER: An Animator/Simulator for Dynamical Systems for IBM Personal Computers: Springer-Verlag, New York, 224 p.

Moon, F.C., 1987, Chaotic Vibrations. An Introduction for Applied Scientists and Engineers: John Wiley and Sons, New York, 309 p.

Percival, I. and Richards, D., 1986, Introduction to Dynamics: Cambridge University Press, 228 p.
One of the few intermediate level introductions to dynamics which includes extensive discussion of nonlinear dynamics.

Schuster, H.G., 1984, [1988 (Second Edition)], Deterministic Chaos: an Introduction: VCH Publishers, Inc., Deerfield Beach, Florida, 220 p.
Perhaps the best introduction after Bergé et al. *(1986).*

Thompson, J.M.T. and Stewart, H.B., 1986, Nonlinear Dynamics and Chaos. Geometric Methods for Engineers and Scientists: John Wiley and Sons, Chichester, 376 p.

Reprint Volumes

Cvitanovic, P., 1984, ed., Universality in Chaos: Adam Hilger Ltd., Bristol, 511 p.
A second printing now available: a good collection of classic papers, with a useful introductory review.

Hao Bai-Lin, 1984, ed., Chaos: World Scientific, Singapore, 576 p.
Includes an introductory review and an extensive bibliography: 40 books and conference proceedings, and about 800 papers.

Geological Applications and Reviews

Chillingworth, D.R.J. and Holmes, P.J., 1980, Dynamical systems and models for reversals of the earth's magnetic field: Mathematical Geology, v. 12, p. 41-59.

Culling, W.E.H., 1987, Equifinality: modern approaches to dynamic systems and their potential for geographic thought: Institution of British Geographers, Transactions, v. 12, no. 1, p. 57-72.

Essex, C., Lookman, T. and Nerenberg, M.R.H., 1987, The climate attractor over short time scales: Nature, v. 326, p. 64-66.

Fowler, A.D., 1988, Geochemical self-organization: Geoscience Canada, v. 15, p. 284-286.

Furbish, D.J., 1988, The river meandering process: evidence for nonlinear chaos: Geological Society of America, Abstracts with Programs, v. 20, p. A362-A363.

Grassberger, P., 1986, Do climatic attractors exist?: Nature, v. 323, p. 609-612.

Henderson, H.W. and Wells, R., 1988, Obtaining attractor dimensions from meteorological time series: Advances in Geophysics, v. 30, p. 205-237.

Ito, K., 1980, Periodicity and chaos in great earthquake occurence: Journal of Geophysical Research, v. 85, p. 1399-1408.

Laskar, J., 1989, A numerical experiment on the chaotic behaviour of the Solar System: Nature, v. 338, p. 237-238.

Moser, J., 1978, Is the solar system stable?: Mathematical Intelligencer, v. 1, p. 65-71.

Nicolis, G. and Baras, F., 1984, eds., Chemical Instabilities: Applications in Chemistry, Engineering, Geology, and Materials Science: D. Reidel Publishing Co., Boston, 420 p.

Nicolis, C. and Nicolis, G., 1984, Is there a climate attractor?: Nature, v. 311, p. 529-532.

Nicolis, C. and Nicolis, G., 1987, eds., Irreversible Phenomena and Dynamical Systems Analysis in Geosciences: D. Reidel Publishing Co., Boston, 578 p.

Scholz, C.H., 1989, Global perspectives of chaos: Nature, v. 338, p. 459-460.

Shaw, H.R., 1987, The periodic structure of the natural record and nonlinear dynamics: Eos, v. 68, p. 1651-1663.

Shaw, H.R., 1988, Mathematical attractor theory and plutonic-volcanic episodicity, *in* Chi-Yu King and Scarpa, R., eds., Modeling of Volcanic Processes: Friedr. Vieweg und Sohn, Braunschweig/Wiesbaden, p. 162-206.

Stewart, C.A. and Turcotte, D.L., 1989, The route to chaos in thermal convection at infinite Prandtle number: 1. Some trajectories and bifurcations: Journal of Geophysical Research, v. 94, p. 13,707-13,717.

Sussman, G.J. and Wisdom, J., 1988, Numerical evidence that the motion of Pluto is chaotic: Science, v. 241, p. 433-437.

Tritton, D.J., 1989, Deterministic chaos, geomagnetic reversals, and the spherical pendulum, *in* Lowes, F.J., et al., eds., Geomagnetism and Paleomagnetism: Kluwer Academic Publishers, Boston, p. 215-226.

Wisdom, J., 1987, Chaotic behaviour in the solar system: Royal Society of London, Transactions, v. A413, p. 109-129.

York, D., 1989, Plotting planet orbits appears impossible over time: The Globe and Mail, Toronto, Ontario, June 17, p. D4.

Other References Cited

Boyce, W.E. and DePrima, R.C., 1986, Elementary Differential Equations, Fourth Edition: John Wiley and Sons, New York, 512 p.

Collet, P. and Eckmann, J.-P., 1980, Iterated Maps on the Interval as Dynamical Systems: Birkhauser, Boston, 248 p.

Grassberger, P. and Procaccia, I., 1983, On the characterization of strange attractors: Physial Review Letters, v. 50, p. 346-349.
See also Grassberger's article in the book by Holden (1986), and references therein.

Greene, J.M., 1986, How a swing behaves: Physica, v. 18D, p. 422-447.

Hénon, M., 1976, A two dimensional mapping with a strange attractor: Communications in Mathematics and Physics, v. 50, p. 69-77.
Describes a particularly simple example of a strange attractor.

Hénon, M. and Heiles, C., 1964, The applicability of the third integral of motion: some numerical experiments: Astronomical Journal, v. 69, p. 73-79.
An early description of a strange attractor, arising from a model based on a problem in galactic motion. The paper was featured as a "citation classic" in Current Contents, *January 25, 1988, p. 18. It has been cited over 500 times. Hénon contributed an interesting brief reminiscence about how the paper came to be written.*

Jain, S.C. and Kennedy, J.F., 1974, The spectral evolution of sedimentary bedforms: Journal of Fluid Mechanics, v. 63, p. 301-314.

Lorenz, E.N., 1963, Deterministic non-periodic flow: Journal of Atmospheric Sciences, v. 20, p. 130-141.
The first strange attractor, though the term itself was first used later, apparently by Ruelle and Takens (1971).

Lorenz, E.N., 1979, On the prevalence of aperiodicity in simple systems, *in* Grmela, M. and Marsden, J., eds., Global Analysis: Springer-Verlag, Berlin, Lecture Notes in Mathematics Series, No. 755, p. 53-75.

May, R.M., 1976, Simple mathematical models with very complicated dynamics: Nature, v. 261, p. 459-467.
Classic discussion of the logistic equation.

Nicolis, G. and Prigogine, I., 1977, Self-Organization in Nonequilibrium Systems. From Disssipative Structures to Order through Fluctutations: John Wiley and Sons, New York, 491 p.

Prüfer, M., 1985, Turbulence in multistep methods for initial value problems: SIAM Journal of Applied Mathematics, v. 45, p. 32-69.
Discusses different finite difference approximations to the logistic equation.

Rössler, O.E., 1976, An equation for continuous chaos: Physics Letters, v. 57A, p. 397-398.

Ruelle, D. and Takens, F., 1971, On the nature of turbulence: Communications in Mathematics and Physics, v. 20, p. 167-192.

Smale, S., 1967, Differentiable dynamical systems: American Mathematical Society, Bulletin, v. 73, p. 747-817.
Discusses the role of stretching and folding in strange attractors: advanced mathematical treatment.

Sparrow, C., 1982, The Lorenz Equations: Bifurcations, Chaos, and Strange Attractors: Springer-Verlag, New York, Applied Mathematics in Science, v. 41, 269 p.

Tai, Sai Choi, 1987, Searching for Chaos in Sunspot Numbers by a Time Series Method, unpublished M.Sc. Thesis, Department of Physics, McMaster University, Hamilton, Ontario.

Tritton, D.J., 1988, Physical Fluid Dynamics, Second Edition: Van Nostrand Reinhold, New York, 519 p.

Yorke, J.A. and Li, Tien-Yien, 1975, Period three implies chaos: American Mathematical Monthly, v. 82, p. 985-992.

Accepted, as revised, 23 January 1990.