



Simplex Adjacency Graphs in Linear Optimization

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Abstract

Simplex Adjacency graphs represent the possible Simplex pivot operations (the edges) between pairs of feasible bases (the nodes) of linear optimization models. These graphs are mainly studied so far in the context of degeneracy. The more general point of view in this paper leads to a number of new results, mainly concerning the connectivity and the feasibility-optimality duality in these graphs. Among others, we present a very short proof of a result of P. Zörnig and T. Gall (1996) on the connectivity of subgraphs corresponding to optimal vertices, and we answer H.-J. Kruse's (1993) question on the connectivity of graphs related to the negative pivots in optimal vertices.

Key words: Simplex Method, Degeneracy Graphs, Linear Programming

1. Introduction

Graphs that have the feasible bases of a given linear optimization model as nodes and the Simplex pivots (both positive and negative) as edges are called in this paper Simplex Adjacency (SA-)graphs. The successive Simplex tableaus that occur during the execution of the Simplex algorithm form a path in such graphs: from an initial feasible basis to an optimal feasible basis (see, for example, Chvátal's graph in Sierksma[13]). SA-graphs are mainly studied in the context of degeneracy (for an extensive account on degeneracy in linear optimization, see Greenberg[10]), where they are called degeneracy graphs; see [2–6,16,8,15,16]. In Gal[2] these graphs are presented for the first time as a tool to study properties of degenerate extreme points of polytopes. This paper collects the interesting properties of SA-graphs, and presents a number of solutions to open problems concerning the connectivity of such graphs.

2. Connectivity in Simplex Adjacency Graphs

We use the constraint collection representation of the feasible region of a given linear optimization model, namely

$$P = \{Ax \leq b; x \geq 0\},$$

with $x \in R^n$, $A \in R^{m \times n}$, $b \in R^m$, ($m, n \geq 1$). It is assumed that P is nonempty. By $pol(P)$ is denoted the

polyhedron in R^n , corresponding to the inequalities in P . A *basis* of P is a subset of $\{x_1, \dots, x_{m+n}\}$ consisting of m elements of which the associated columns in $[AI_m]$ are independent, with x_1, \dots, x_n the entries of x and x_{n+1}, \dots, x_{m+n} the slack variables of $Ax \leq b$. The *Simplex Adjacency (SA-) graph* of P , notation $SA(P)$, is a graph of which the nodes are the feasible bases of P , and two nodes are connected by an edge iff the two associated bases can be obtained from each other by one single Simplex pivot operation.

Every feasible basis of P corresponds to a vertex of $pol(P)$, but a vertex of $pol(P)$ may correspond to several feasible bases, and hence to several nodes of $SA(P)$. A vertex that is k -degenerate (see e.g. Sierksma,Tijssen[14]) therefore corresponds to $\binom{n}{k}$ nodes. An edge of $SA(P)$ is called a *positive (negative) edge* if the associated pivot is performed on a positive (negative) entry in the associated Simplex tableau. In this paper we represent Simplex tableaus in the form of the convenient Tucker Simplex tableaus; see e.g. Balinski,Tucker[1], and Tijssen,Sierksma[14]. The subgraph of $SA(P)$ that contains only the positive (negative) edges of $SA(P)$ is called the *positive(negative) SA-graph*, and is denoted by $SA(P)_+$ ($SA(P)_-$). In Theorem 2, it will be shown that $SA(P)$ is connected for each P . However, both $SA(P)_+$ and $SA(P)_-$ may be disconnected.

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In general, if $pol(P)$ contains at least two vertices,

then $SA(P)_-$ is disconnected, since it is not possible to move from one feasible basis in one vertex to another feasible basis in another vertex by negative pivots only. Therefore, if $SA(P)_-$ is connected, $pol(P)$ contains only one vertex. However, this is not sufficient for the connectivity of $SA(P)_-$, as the following example shows.

Consider the constraint collection:

$$P = \{2x_1 \leq 0; x_1 \geq 0\},$$

and use x_2 as the slack variable of the first constraint. Then $pol(P)$ has only one vertex ($x_1 = x_2 = 0$), and two bases B_1 and B_2 that are connected in the SA-graph of P by an edge that corresponds to a positive Simplex pivot. The two corresponding Tucker Simplex tableaus then read (with rhs = right hand side):

$$\begin{array}{c}
 \begin{array}{cc}
 \text{rhs} & x_1 \\
 g = \begin{array}{|c|c|} \hline 0 & 0 \\ \hline 0 & 2 \\ \hline \end{array} & \\
 -x_2 = & \\
 \text{basis } B_1 : \{x_2\} &
 \end{array}
 \quad
 \begin{array}{cc}
 \text{rhs} & x_2 \\
 g = \begin{array}{|c|c|} \hline 0 & 0 \\ \hline 0 & 0.5 \\ \hline \end{array} & \\
 -x_1 = & \\
 \text{basis } B_2 : \{x_1\} &
 \end{array}
 \end{array}$$

The negative SA-graph $SA(P)_-$ of P consists of the two vertices of B_1 and B_2 and contains no edges. Therefore, $SA(P)_-$ is not connected. In general, positive SA-graphs are connected. If no vertex of $pol(P)$ is degenerate, every feasible basis can be reached from any other feasible basis by means of positive pivots. This is the usual case when the Simplex algorithm is applied. It is however possible that a positive SA-graph $SA(P)_+$ is not connected; see the example below. First we give a number of properties of a constraint collection P with a disconnected positive SA-graph. We use the following concept. Let v be any vertex of $pol(P)$. The *Simplex v -Adjacency graph*, denoted by $SA(P)^v$, is the subgraph of $SA(P)$ induced by the feasible bases of v . The subgraph of $SA(P)^v$ that contains the positive (negative) edges is denoted by $SA(P)_+^v$ ($SA(P)_-^v$), respectively.

Theorem 1 *Let $SA(P)_+$ be a disconnected positive SA-graph of the constraint collection P . Then for each vertex v of $pol(P)$, the following assertions hold.*

- (a) *Each connected component of $SA(P)_+^v$ contains a node corresponding to v ;*
- (b) *$SA(P)_+^v$ is disconnected, and every vertex of $pol(P)$ is degenerate;*
- (c) *$pol(P)$ is unbounded.*

Proof. (a) Let CC be a connected component of $SA(P)_+$, and let $v \in CC$. Construct a Tucker Simplex

tableau that corresponds to v . Then, take any vertex of $pol(P)$, and choose an objective function for which this vertex is optimal. Use a Simplex pivot rule, that uses only positive pivots to solve this tableau. The Simplex method creates a path in CC that connects the selected node with a basis corresponding to v . Hence, the arbitrary selected vertex v corresponds to a node (basis) of CC .

(b) Take any vertex v of $pol(P)$. Since $SA(P)_+$ is disconnected, and every component of $SA(P)_+$ contains a node that corresponds to a basis of v , $SA(P)_+^v$ is also disconnected. Therefore, $SA(P)_+^v$ contains at least two nodes. Hence v is degenerate.

(c) Take any vertex v of $pol(P)$, and let B_1 and B_2 be two bases of v from different components of $SA(P)_+$. The basic variables in v , that have a strictly positive value, are in both bases, so that the difference between the two bases concerns only the basic variables with zero values. Define $Q = B_1 \setminus B_2$, and $R = B_2 \setminus B_1$; i.e. $Q \cup R$ is the symmetric difference of B_1 and B_2 . We assume Q and R to be as small as possible; i.e. with a positive pivot it is not possible to make Q and R smaller. We will show that the Tucker Simplex tableau corresponding to B_1 contains a column without positive entries, but with at least one negative entry, meaning that $pol(P)$ is in fact unbounded. Let A_1 be the sub matrix in the tableau of B_1 of which the columns correspond to the nonbasic variables in R , and the rows to the basic variables in Q . Since B_1 and B_2 are both feasible bases, A_1 is a nonsingular sub matrix, and because Q and R are as small as possible, A_1 has no positive entries, and every column of A_1 contains at least one nonzero entry. Let x_r be a nonbasic variable w.r.t. B_1 in R , and let x_q be a basic variable in Q that corresponds to a negative entry in the column of x_r in A_1 . Pivoting on this negative entry results in the basis $(B_1 \setminus \{x_q\}) \cup \{x_r\}$, which is a feasible basis, since the value of x_q is zero. If the column of x_r does not contain a positive entry, then the value of x_r can be increased unlimitedly, and therefore $pol(P)$ is unbounded. If, on the other hand, the column of x_r contains a positive entry, we can find a basic variable x_z , by means of a usual ratio test, such that a pivot on this positive entry yields the feasible basis $(B_1 \setminus \{x_z\}) \cup \{x_r\}$. After this pivot, the entry on the intersection of the column of x_z and the row of x_q has become positive. Pivoting on this positive entry results in the basis $((B_1 \setminus \{x_z\}) \cup \{x_r\}) \setminus \{x_q\} \cup \{x_z\} = (B_1 \setminus \{x_q\}) \cup \{x_r\}$, which is again a feasible basis. Since the difference between this basis and B_2 is

smaller than the difference between B_1 and B_2 , we have a contradiction with the assumption that the difference between B_1 and B_2 is as small as possible. Therefore, the column of x_r cannot contain a positive entry, and hence $pol(P)$ is unbounded. ■

We will illustrate Theorem 1 by means of a small example. Consider the following constraint collection:

$$P = \{-x_1 \leq 0; -2x_1 + x_2 + x_3 \leq 1; x_1, x_2, x_3 \geq 0\}.$$

The possible Simplex tableaus are (using x_4 and x_5 as slack variables):

	rhs	x_1	x_2	x_3
$g =$	0	0	0	0
$-x_4 =$	0	-1	0	0
$-x_5 =$	-1	-2	1	1
basis $B_1 : \{x_4, x_5\}$				

	rhs	x_1	x_2	x_5
$g =$	0	0	0	0
$-x_4 =$	0	-1	0	0
$-x_3 =$	-1	-2	1	1
basis $B_2 : \{x_4, x_3\}$				

	rhs	x_1	x_5	x_3
$g =$	0	0	0	0
$-x_4 =$	0	-1	0	0
$-x_2 =$	-1	-2	1	1
basis $B_3 : \{x_4, x_2\}$				

	rhs	x_4	x_2	x_3
$g =$	0	0	0	0
$-x_1 =$	0	-1	0	0
$-x_5 =$	-1	-2	1	1
basis $B_4 : \{x_1, x_5\}$				

	rhs	x_4	x_2	x_5
$g =$	0	0	0	0
$-x_1 =$	0	-1	0	0
$-x_3 =$	-1	-2	1	1
basis $B_5 : \{x_1, x_3\}$				

	rhs	x_4	x_5	x_3
$g =$	0	0	0	0
$-x_1 =$	0	-1	0	0
$-x_2 =$	-1	-2	1	1
basis $B_6 : \{x_1, x_2\}$				

The Simplex Adjacency graph of this example is depicted in Figure 1.

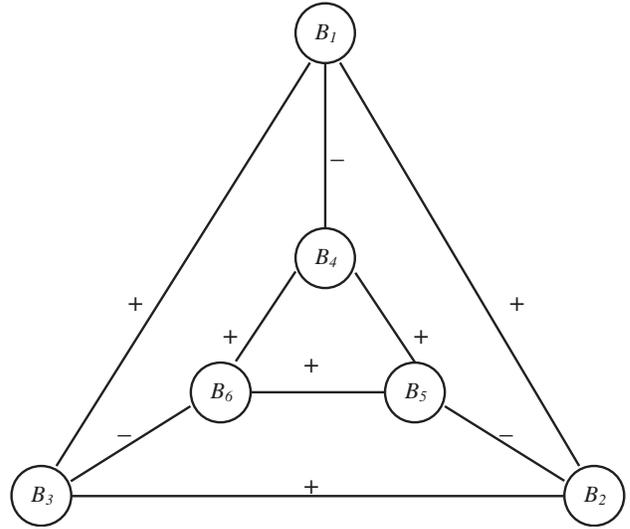


Fig. 1. Simplex Adjacency Graph.

The vertices of the feasible region from Figure 1 satisfy: $(x_1 = x_2 = x_3 = x_4 = 0, x_5 = 1)$, $(x_1 = x_2 = x_4 = x_5 = 0, x_3 = 1)$, and $(x_1 = x_3 = x_4 = x_5 = 0, x_2 = 1)$. It is easy to see that every vertex contains two different bases and is therefore degenerate. Namely, B_1 and B_4 correspond to $(x_1 = x_2 = x_3 = x_4 = 0, x_5 = 1)$, B_2 and B_5 correspond to $(x_1 = x_2 = x_4 = x_5 = 0, x_3 = 1)$, and B_3 and B_6 correspond to $(x_1 = x_3 = x_4 = x_5 = 0, x_2 = 1)$. Furthermore, $pol(P)$ is unbounded, because the column of x_1 in the tableau of B_1 contains no positive entry. Note that the positive SA-graph consists of two disconnected triangles (B_1, B_2, B_3) and (B_4, B_5, B_6) .

The following theorem on the connectivity of SA-graphs can also be found in Kruse[12].

Theorem 2 . *The Simplex Adjacency graph of any constraint collection is connected.*

Proof. Let $SA(P)$ be the Simplex Adjacency graph of the constraint collection P . In the trivial case that $SA(P)$ contains only one feasible basis, the theorem is obviously true. Assume that there exist two different feasible bases, say B_1 and B_2 . Start with a Tucker Simplex tableau with B_1 as basis and an objective function that is optimal for B_2 . Use a Simplex pivot rule, that uses only positive pivots to solve this tableau. The Simplex algorithm follows a path in $SA(P)_+$ that connects B_1 with another basis, say B_3 , such that B_2 and B_3 correspond to the same vertex. Similar as in the proof of

Theorem 1, the matrix A_1 can be constructed from the current tableau of B_3 , namely take the intersection of the rows and columns that correspond to the symmetric difference of B_2 and B_3 . By executing pivot operations on (positive or negative) entries of A_1 , the symmetric difference of B_2 and the current B_3 can be made smaller; the process stops when the current B_3 equals B_2 . In this way a path from B_1 to B_2 is constructed. Hence, $SA(P)$ is connected. ■

3. Optimality in Simplex Adjacency Graphs

Consider the linear optimization model:

$$\max\{c^T x \mid Ax \leq b; x \geq 0\},$$

with $A \in R^{m \times n}$, $b \in R^m$, $c, x \in R^n$. It is assumed that this model has an optimal solution. As usual, $P = \{Ax \leq b; x \geq 0\}$. For any optimal vertex v of $pol(P)$, the *optimal vSA-graph*, $OptSA(P)^v$, is the subgraph $SA(P)^v$ induced by the optimal feasible bases corresponding to v . The subgraph of $OptSA(P)^v$ that contains the positive (resp. negative) edges is called the *positive* (resp. *negative*) optimal vSA-graph, and is denoted by $OptSA(P)^v_+$ (resp. $OptSA(P)^v_-$). A natural question is: for which constraint collections P is the graph corresponding to all feasible bases of v equal to the graph corresponding to only the optimal feasible bases of v , i.e.

$$SA(P)^v = OptSA(P)^v?$$

In Kruse[12], it is shown that it may happen that all basic feasible solutions of a degenerate vertex are optimal. Kruse also gives an example of a linear optimization model where the optimal vertex has a degeneracy degree of one, and reports that examples with degeneracy degrees larger than one are not known. He even conjectures that the latter case may never occur. Below we show that Kruse's conjecture is wrong. It can simply be shown by means of the following example. This example even shows that optimal vertices, of which the degeneracy degree has an arbitrary value, exist. Consider the following linear optimization model:

$$\begin{aligned} \min \quad & y \\ \text{s.t.} \quad & Ax \leq 0 \\ & y \leq 1 \\ & x \geq 0 \\ & y \geq 0 \end{aligned}$$

with $A \in R^{m \times n}$, $x \in R^m$, and $y \in R$. Obviously, the vertex with $(y = 0, x = 0)$ is optimal. Pivoting in A does not change the feasibility nor the optimality. A pivot in the column corresponding to y destroys the optimality. Therefore, all feasible bases corresponding to $(x = 0, y = 0)$ are optimal. The degeneracy degree of this vertex is m .

In the following theorem we show that, using duality between optimality and feasibility, any optimal SA-graph is isomorphic to (\cong) some SA-graph.

Theorem 3 Consider a linear optimization model of which v is an optimal vertex and $pol(P)$ is the feasible region. Then there exists a constraint collection P' such that

$$OptSA(P)^v \cong SA(P').$$

Moreover, $OptSA(P)^v_+ \cong SA(P')_-$, and $OptSA(P)^v_- \cong SA(P')_+$.

Proof. Let T be the Tucker Simplex tableau corresponding to $OptSA(P)^v$. Positive pivots in rows of T that correspond to basic variables with a positive optimal value are not allowed, since such pivots cause the 'leaving' of the optimal vertex. So, these rows can be deleted from T . We then obtain a reduced tableau, say rT , that still corresponds $OptSA(P)^v$. The linear optimization model that corresponds to rT is called model rT . This model consists only of the constraints that are binding at v , and represents a convex cone with apex v . The negative transpose of rT corresponds to a dual model of which the SA-graph is isomorphic to the optimal SA-graph of the model rT . Every pivot on a positive entry in the dual tableau transforms it into a tableau that is the negative transpose of the resulting rT tableau after a pivot on the corresponding positive entry in the rT tableau. Therefore, the positive optimal SA-graph of model rT is isomorphic to the negative SA-graph of the dual model, and the negative optimal SA-graph of model rT is isomorphic to the positive SA-graph of the dual model. ■

Let P be a constraint collection of some linear optimization model. A constraint collection P' from Theorem 3 can be constructed as follows. Let P^v be the collection of constraints of P that are binding at the optimal vertex v . Note that these constraints form a cone with apex v . We dualize the model with constraint collection P^v , instead of P . This dual constraint collection has a SA-graph isomorphic to $SA(P)^v$. We

will illustrate the construction of such a constraint collection with the following example.

Consider the following linear optimization model:

$$\begin{array}{ll}
 \max & -4x_4 \\
 \text{s.t.} & -x_4 \leq 0 \text{ (slack : } x_5) \\
 & -2x_2 - x_3 + 4x_4 \leq 0 \text{ (slack : } x_6) \\
 -x_1 - x_2 + x_3 + x_4 & \leq 0 \text{ (slack : } x_7) \\
 & +x_2 - 3x_3 \leq 2 \text{ (slack : } x_8) \\
 3x_1 + 2x_2 - 5x_3 - 5x_4 & \leq 3 \text{ (slack : } x_9) \\
 x_1, x_2, x_3, x_4 & \geq 0.
 \end{array}$$

The corresponding optimal Tucker Simplex tableau is:

	1	x_1	x_2	x_3	x_4
$-f =$	0	0	0	0	4
$-x_5 =$	0	0	0	0	-1
$-x_6 =$	0	0	-2	-1	4
$-x_7 =$	0	-1	-1	1	1
$-x_8 =$	-2	0	1	-3	0
$-x_9 =$	-3	3	2	-5	-5

In the optimal vertex, corresponding to this tableau, only the basic variables x_8 and x_9 have a strictly positive value. After removing the rows of x_8 and x_9 , and taking the negative transpose of the new problem, we obtain the following optimal dual tableau.

	1	y_5	y_6	y_7
$-g =$	0	0	0	0
$-y_1 =$	0	0	0	1
$-y_2 =$	0	0	2	1
$-y_3 =$	0	0	1	-1
$-y_4 =$	-4	1	-4	-1

The corresponding constraint collection of this (dual) model reads:

$$\begin{array}{l}
 y_7 \leq 0 \text{ (slack : } y_1) \\
 2y_6 + y_7 \leq 0 \text{ (slack : } y_2) \\
 y_6 - y_7 \leq 0 \text{ (slack : } y_3) \\
 y_5 - 4y_6 - y_7 \leq 4 \text{ (slack : } y_4) \\
 y_5, y_6, y_7 \geq 0,
 \end{array}$$

which has a SA-graph that is isomorphic to the optimal SA-graph of the original linear optimization model.

Theorem 3 immediately implies the following. First, optimal SA-graphs are connected, since all SA-graphs are connected (see Theorem 2). A proof of this fact is also given in Zörnig and Gal[16]. Secondly, positive optimal SA-graphs are in general disconnected, except

if all bases belong to one dual vertex, since negative SA-graphs are in general disconnected, except if all bases belong to the same vertex.

A further question asked in Kruse[12] is whether lack of full dimensionality is a necessary condition for the disconnectedness of negative optimal SA-graphs. This question is answered in the following theorem.

Theorem 4 *If $\text{OptSA}(P)_-^v$ is disconnected for some optimal vertex v of $\text{pol}(P)$, then $\text{pol}(P)$ is not fully dimensional.*

Proof. According to Theorem 3, a negative optimal SA-graph is isomorphic to a positive SA-graph. Theorem 1 shows that if a positive SA-graph is disconnected, its corresponding polyhedron is unbounded. For this polyhedron there is a Tucker Simplex tableau that has a column without any positive entry, and at least one negative entry. Taking again the negative transpose of this tableau, we obtain a tableau of the initial model that has a row with nonnegative entries. This means that the values of some variables are zero for every feasible point, and hence, the feasible region is not fully dimensional. ■

Simplex Adjacency graphs can be seen as special cases of optimal SA-graphs in which all bases in a specified optimal vertex are optimal. This can be accomplished by taking an all-zero objective function in a degenerate vertex. This is formulated in the following theorem.

Theorem 5 *Let P be the constraint collection of some linear optimization model. Then for each vertex v of $\text{pol}(P)$, there is a constraint collection P' such that $\text{SA}(P') \cong \text{SA}(P)^v$; moreover, $\text{SA}(P)_+^v \cong \text{SA}(P')_-$, and $\text{SA}(P)_-^v \cong \text{SA}(P')_+$.*

Proof. The proof is similar to the proof of Theorem 3. We use an all-zero objective function. Therefore, the objective row in an optimal tableau will have zero entries, as well as the right hand side in the negative transpose. ■

So far we have constructed SA-graphs from a given constraint collection. In general, we call a graph G a SA-graph if there exists a constraint collection P such that $G \cong \text{SA}(P)$. Similar definitions can be given for vSA-graphs and OptSA-graphs.

Theorem 6 *The three classes consisting of, respectively, the SA-graphs, the vSA-graphs, and the OptSA-*

graphs are pair wise isomorphic.

Proof. This result follows immediately from Theorem 3 and Theorem 5. ■

Since the classes of SA-, vSA-, and OptSA-graphs are pair wise isomorphic, all properties of SA-graphs can easily be translated to properties of vSA-graphs, and of OptSA-graphs as well. In order to translate a property of a SA-graph into a similar property of a vSA-graph or an OptSA-graph, it suffices to construct the ‘negative transpose’ of that property.

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