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On-line bin packing with two item sizes

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Abstract

We study the on-line bin packing problem (BPP). In BPP, we are given a sequence B of items $a_1, a_2, \ldots, a_n$ and a sequence of their sizes $(s_1, s_2, \ldots, s_n)$ (each size $s_i \in (0,1]$) and are required to pack the items into a minimum number of unit-capacity bins. Let $R^\infty_{\{\alpha,\beta\}}$ be the minimal asymptotic competitive ratio of an on-line algorithm in the case when all items are only of two different sizes α and β. We prove that $\max\{R^\infty_{\{\alpha,\beta\}}:\ \alpha,\beta \in (0,1]\} = 4/3$. We also obtain an exact formula for $R^\infty_{\{\alpha,\beta\}}$ when $\max\{\alpha,\beta\} > \frac{1}{2}$. This result extends the result of Faigle, Kern and Turan (1989) that $R^\infty_{\{\alpha,\beta\}} = \frac{4}{3}$ for $\beta = \frac{1}{2} - \epsilon$ and $\alpha = \frac{1}{2} + \epsilon$ for any fixed nonnegative $\epsilon < \frac{1}{6}$.

Key words: On-line algorithms, bin packing, competitive ratio.

1. Introduction

In this paper we study the classical on-line bin packing problem (BPP), which is one of the oldest and most well-studied problems in optimization. In BPP, we are given a sequence B of items $a_1, a_2, \ldots, a_n$ and a sequence of their sizes $(s_1, s_2, \ldots, s_n)$ (each size $s_i \in (0,1]$) and are required to pack the items into a minimum number of unit-capacity bins. In other words, we need to partition B into a minimum number m of subsets $B_1, B_2, \ldots, B_m$ so that $\sum_{a_i \in B_j} s_i \le 1$ for each $j = 1, 2, \ldots, m$. For surveys of BPP, see [3–5].

For any $S \subseteq (0,1]$, we let $\mathcal{B}(S)$ denote the set of all sequences B with all item sizes $s_i \in S$, $i = 1, 2, \ldots, n$. For a given sequence L and an on-line algorithm A, let $A(L)$ be the number of bins required for L by algorithm A; let $\mathrm{OPT}(L)$ be the minimum number of bins needed to pack the items of L off-line, that is, when they are all available at once. The *asymptotic competitive ratio* $R^\infty_S(A)$ of A on $\mathcal{B}(S)$ is

$$\limsup_{N\to\infty} \max\{\tfrac{A(L)}{\mathrm{OPT}(L)}:\ L \in \mathcal{B}(S), \mathrm{OPT}(L) = N\}.$$

With $S = (0,1]$ we note that $R^\infty_S(A) = R^\infty(A)$ is the usual asymptotic competitive ratio of an on-line bin packing algorithm A. Let R^∞_S be the minimum possible asymptotic competitive ratio of an algorithm for the bin packing problem on $\mathcal{B}(S)$. An on-line algorithm A with $R^\infty_S(A) = R^\infty_S$ is called an *optimal* algorithm.

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Ullman [11] was the first to investigate the on-line bin packing problem. He proved that the FIRST FIT algorithm has asymptotic competitive ratio 1.7. This result was then published in [7]. Yao [12] showed that REVISED FIRST FIT has asymptotic competitive ratio $\frac{5}{3}$ and proved that every on-line BPP algorithm has asymptotic competitive ratio at least 1.5. Yao's upper bound was improved by Seiden [9] to 1.58889, which is currently the best result. Brown [1] and Liang [8] independently improved Yao's lower bound to 1.53635. This was further improved by van Vliet [10] to 1.54014. Chandra [2] showed that the preceding lower bounds also apply to randomized algorithms. So, currently no optimal on-line BPP algorithm is known.

In many applications of BPP, there is only a small number of item sizes and, thus, it makes sense to study on-line algorithms specialized to pack inputs from $\mathcal{B}(S)$, where S is a small set of item sizes.

In this paper, we study $R^\infty_{\{\alpha,\beta\}}$, where $\alpha,\beta \in (0,1]$. Our main result is that $\max\{R^\infty_{\{\alpha,\beta\}}:\ \alpha,\beta \in (0,1]\} = 4/3$ (see Theorem 9). The easy lower bound $\max\{R^\infty_{\{\alpha,\beta\}}:\ \alpha,\beta \in (0,1]\} \ge 4/3$ was shown in [8,12] (see also Lemma 1 in this paper) and we prove the matching upper bound is a series of lemmas.

We also study $R^\infty_{\{\alpha,\beta\}}$ in more detail for the case $\max\{\alpha,\beta\} > \frac{1}{2}$. In Theorem 3, we obtain $R^\infty_{\{\alpha,\beta\}}$ for all values of α and β provided $\max\{\alpha,\beta\} > \frac{1}{2}$. Our result extends the result of Faigle, Kern and Turan [6] that $R^\infty_{\{\alpha,\beta\}} = 4/3$ for $\beta = \frac{1}{2} - \epsilon$ and $\alpha = \frac{1}{2} + \epsilon$ for every fixed nonnegative $\epsilon < \frac{1}{6}$ (see Theorem 9 in [6]).

It seems much harder to obtain an exact formula for $R^{\infty}_{\{\alpha,\beta\}}$ for all values of α and β. Also, for $t \geq 3$ and BPP on $\mathcal{B}(S)$ with $|S| \leq t$, it seems much more difficult to design optimal on-line algorithms. We believe that these problems are worth studying from both theoretical and practical points of view.

In what follows, let α and β denote the two item sizes, where $0 < \beta \leq \alpha \leq 1$. For simplicity we will denote items of size α by α-items, and items of size β by β-items. We assume that all bins have capacity 1. Let x_i denote the largest integer such that $x_i\beta + i\alpha \leq 1$ (i.e. at most x_i items of size β will fit in a bin together with i items of size α). We say that *almost all* bins of a set S satisfy a certain property if all or all except one bin in S satisfy this property.

2. When $\alpha > \frac{1}{2}$

We start by proving a lower bound for $R^{\infty}_{\{\alpha,\beta\}}$.

Lemma 1 *If $\alpha > 1/2$, then*

$$R^{\infty}_{\{\alpha,\beta\}} \geq \frac{x_0^2}{x_0^2 - x_1(x_0 - x_1)} \geq \frac{4}{3}.$$

PROOF. If $x_1 = 0$ then the lemma clearly holds as in this case $\frac{x_0^2}{x_0^2 - x_1(x_0-x_1)} = 1$, so we may assume that $x_1 > 0$. Clearly we also have $x_0 > x_1$.

Let A be an optimal algorithm for the given α and β, and assume that the input starts of with k β-items. Let B_{full} be the number of bins produced by A, which do not have space for an additional α-item, and let B_α be the number of bins where an α-item would still fit. Since A has asymptotic competitive ratio $R^{\infty}_{\{\alpha,\beta\}}$, the following holds, for some constant c^*, not depending on k.

$$B_{full} + B_\alpha - c^* \leq R^{\infty}_{\{\alpha,\beta\}} \frac{k}{x_0}$$

Furthermore if another $\frac{k}{x_1}$ α-items arrive after the k β-items, then the following must also hold.

$$B_{full} + \frac{k}{x_1} - c^* \leq R^{\infty}_{\{\alpha,\beta\}} \frac{k}{x_1}$$

Observe that the above two inequalities are equivalent to the following:

$$x_0 \frac{B_{full} + B_\alpha}{k} \leq R^{\infty}_{\{\alpha,\beta\}} + \frac{c^* x_0}{k},$$
$$x_1 \frac{B_{full}}{k} + 1 \leq R^{\infty}_{\{\alpha,\beta\}} + \frac{c^* x_1}{k}$$

Let $r'_k = R^{\infty}_{\{\alpha,\beta\}} + \frac{c^* x_0}{k}$ and observe that r'_k tends to $R^{\infty}_{\{\alpha,\beta\}}$ when k goes to infinity, as c^* and x_0 do not depend on k. Furthermore, since $k \leq x_0 B_{full} + x_1 B_\alpha$ and $x_0 > x_1$ we conclude that the following must hold.

$$x_0 \frac{B_{full} + B_\alpha}{x_0 B_{full} + x_1 B_\alpha} \leq r'_k,$$
$$x_1 \frac{B_{full}}{x_0 B_{full} + x_1 B_\alpha} + 1 \leq r'_k$$

Let $B = \frac{B_{full}}{B_\alpha}$ and let $\gamma = \frac{x_1}{x_0}$. The above inequalities can be rewritten as follows:

$$f(B) \leq r'_k, \qquad \text{where } f(t) = \frac{t+1}{t+\gamma}$$
$$g(B) \leq r'_k, \qquad \text{where } g(t) = \frac{\gamma t}{t+\gamma} + 1$$

Since γ is a constant and $0 < \gamma < 1$, observe that $f(t)$ is a decreasing function and $g(t)$ is an increasing function. Thus, if $f(t_0) = g(t_0)$, then $f(t_0) \leq \max\{f(B), g(B)\} \leq r'_k$.

This implies the following:

$$\begin{aligned} r'_k \geq g(\frac{1-\gamma}{\gamma}) = f(\frac{1-\gamma}{\gamma}) &= \frac{(1-\gamma)/\gamma + 1}{(1-\gamma)/\gamma + \gamma} \\ &= \frac{1}{1-\gamma(1-\gamma)} = \frac{1}{1 - \frac{x_1}{x_0}(1 - \frac{x_1}{x_0})} \\ &= \frac{x_0^2}{x_0^2 - x_1(x_0 - x_1)} \end{aligned}$$

As mentioned earlier r'_k tends to $R^{\infty}_{\{\alpha,\beta\}}$ when k goes to infinity, which implies that $R^{\infty}_{\{\alpha,\beta\}} \geq \frac{x_0^2}{x_0^2 - x_1(x_0-x_1)}$. By $(x_0 - 2x_1)^2 \geq 0$, we have $\frac{x_0^2}{x_0^2 - x_1(x_0-x_1)} \geq \frac{4}{3}$.

Lemma 2 provides the matching upper bound for $R^{\infty}_{\{\alpha,\beta\}}$, and its proof consists in exhibiting an algorithm for this problem and a suitable upper bound for its asymptotic performance ratio. The difficulty to overcome when packing items of sizes $\alpha > 1/2$ and $\beta < 1/2$ on-line is to keep a proper balance between the numbers of those bins which become packed full solely with β-items, and those bins which are packed so as to have enough room left for an α-item in addition to any β-items. Accumulating too many bins of the former type during the packing procedure will be harmful if subsequent input items turn out to be all α-items, each of which will need to be put in an additional bin, thus leaving the solution far from optimal. Whereas a surplus of bins of the latter type leaves the solution suboptimal in the event that no new input items arrive. Thus throughout execution of the algorithm described in the following proof, the primary objective is to distribute the arriving β-items so as to keep close to a certain ratio Q between the two types of bins at all times, where Q depends on the sizes α and β. With this in mind, the algorithm is fairly straightforward.

Corollary 2 *If $\alpha > 1/2$, then $R^\infty_{\{\alpha,\beta\}} \leq \frac{x_0^2}{x_0^2 - x_1(x_0-x_1)}$.*

PROOF. If $x_1 = 0$ then it is not difficult to obtain $R^\infty_{\{\alpha,\beta\}} = 1$, so assume that $x_1 > 0$. Clearly we also have $x_0 > x_1$.

Let $Q = \frac{x_1}{x_0 - x_1}$, and consider the following on-line algorithm.

Our algorithm will maintain four sets of bins, A_0, A_1, B_0 and B_1. They will have the following properties.

A_0 consists of bins with 0 α-items and at least 1 β-item

A_1 consists of bins with 1 α-item and at least 1 β-item

B_0 consists of bins with 0 α-items and at most x_1 β-items

B_1 consists of bins with 1 α-item and 0 β-items

The bins in A_0 are committed to being filled entirely with β-items. All other β-items that arrive will be distributed in bins from A_1 and B_0, where they, respectively, either join an already packed α-item, or wait for an α-item to arrive and be packed into the same bin. The bins in B_1 are used only in case of a momentary surplus of α-items.

Let a_0, a_1, b_0 and b_1 denote the number of bins in A_0, A_1, B_0 and B_1, respectively. The algorithm proceeds by the following guidelines:

- If the next item is an α-item, and $b_0 > 0$, then add the item to a bin in B_0, and move the resulting bin from B_0 to A_1. If $b_0 = 0$ then put the α-item in a new bin, and add it to B_1.
- If the next item is a β-item, then apply one of the following rules, listed in order of priority:
  - · If a bin in A_1 does not contain x_1 β-items, then add the β-item to this bin.
  - · If a bin in A_0 does not contain x_0 β-items, then add the β-item to this bin.
  - · If a bin in B_0 does not contain x_1 β-items, then add the β-item to this bin.
  - · If $b_1 > 0$, then add the β-item to a bin in B_1, and move the bin to A_1.
  - · Otherwise add the β-item to a new bin. If $\frac{b_0+a_1}{a_0+1} < Q$ then add the new bin to B_0, and if $\frac{b_0+a_1}{a_0+1} \geq Q$ then add the new bin to A_0.

Observe that $a_0 = b_0 = a_1 = 0$ if and only if the input does not have β-items, in which case the algorithm produces the optimal solution. Thus, in the rest of the proof, we may assume that at least one β-item is present in the input, so $\max\{a_0, b_0, a_1\} > 0$.

During the entire execution of the algorithm, the following properties hold.

(a): Almost all bins in A_0 contain x_0 β-items.

(b): Almost all bins in A_1 and B_0 contain x_1 β-items.

(c): $b_0 = 0$ or $b_1 = 0$. This is clearly true in the beginning of the algorithm and one can see from the algorithm description that b_i becomes positive only if $b_{1-i} = 0$ for $i = 0, 1$.

(d): $\frac{b_0+a_1}{a_0} \geq Q$ (for $a_0 = 0$, $\frac{b_0+a_1}{a_0} = \infty$ since $\max\{a_0, b_0, a_1\} > 0$, so we may assume that $a_0 > 0$). Indeed, the operations of the algorithm, except the very last one, do not decrease the value of $\frac{b_0+a_1}{a_0}$. The last operation only decreases the value of $\frac{b_0+a_1}{a_0}$ by increasing the value of a_0 by 1 if $\frac{b_0+a_1}{a_0+1} \geq Q$. So even after decreasing the value of the fraction $\frac{b_0+a_1}{a_0}$, it remains at least Q. Moreover, just after a_0 turns from zero to one, $\frac{b_0+a_1}{a_0} \geq Q$ by the corresponding condition of the last operation of the algorithm.

(e): $\frac{b_0+a_1-1}{a_0+1} < Q$. This is the case as the fraction $\frac{b_0+a_1-1}{a_0+1}$ equals -1 in the beginning of the algorithm and can only increase if a β-item arrives and one of the last two rules is applied. However, by (c), since the most recent increase of b_0 (using the last rule), the second last operation is no longer used. Notice that at the time when b_0 was last increased we had $\frac{b_0+a_1}{a_0+1} < Q$.

Let $r = \frac{x_0^2}{x_0^2 - x_1(x_0-x_1)}$.

(f): If $b_1 > 0$, then $a_1(r-1) + a_0(\frac{(r-1)x_0}{x_1} - 1) \geq 0$. This holds by the following:

$$
\begin{aligned}
& a_1(r-1) + a_0(\frac{(r-1)x_0}{x_1} - 1) \\
& \quad = a_0(r-1)[\frac{a_1}{a_0} + (\frac{x_0}{x_1} - \frac{1}{r-1})] \\
& \quad \geq a_0(r-1) \\
& \qquad\quad [Q + \frac{x_0(x_0-x_1)}{x_1(x_0-x_1)} - \frac{x_0^2 - x_1(x_0-x_1)}{x_1(x_0-x_1)}] \\
& \quad = a_0(r-1)[\frac{x_1^2}{x_1(x_0-x_1)} + \frac{-x_1^2}{x_1(x_0-x_1)}] \\
& \quad = 0
\end{aligned}
$$

In the above argument, we use the inequality $Q \leq a_1/a_0$, which holds due to (c) and (d).

Since we are only considering asymptotic ratios, we may by (a) and (b) assume that all bins in B_0 and A_1 contain x_1 β-items, and all bins in A_0 contain x_0 β-items. By (d) and (e), $\frac{b_0+a_1-1}{a_0+1} < Q \leq \frac{b_0+a_1}{a_0}$. Define $a'_1 = a_1 - e$, $a'_0 = a_0 + e$, and $f(e) = (b_0 + a'_1)/a'_0$ for $e \in [0, 1]$.

We have

$$f(1) = \tfrac{b_0+a_1-1}{a_0+1} \leq f(e) \leq \tfrac{b_0+a_1}{a_0} = f(0).$$

Since $f(e)$ is a continuous function, for some value of $e \in [0, 1)$, $\frac{b_0+a'_1}{a'_0} = Q$. Let us fix these values of a'_0 and a'_1.

Let *opt* denote the size of the optimal solution. We are now ready to prove that $r \cdot opt - (a_0 + a_1 + b_0 + b_1) \geq 0$, which would complete the proof. Before considering the following four cases, let us estimate the value of *opt*. Since each α-item requires a separate bin, the optimum equals $a_1 + b_1$ plus γ, the minimum number of bins required to accommodate the β-items not fitting into the bins from A_1 and B_1. Taking into consideration that $a_1 + b_1$ bins with α-items in them may accommodate up to $(a_1 + b_1)x_1$ β-items,

$$\gamma \leq \max\{0, (a_1x_1 + a_0x_0 + b_0x_1 - (a_1 + b_1)x_1)/x_0\}.$$

Thus,

$$opt \geq a_1 + b_1 + \max\{0, (a_0x_0 + b_0x_1 - b_1x_1)/x_0\}. \quad (1)$$

Case 1. $b_0 > 0$**:** By (c), we observe that $b_1 = 0$ and, by (1) and $a_0 + a_1 = a'_0 + a'_1$, we get the following:

$$\begin{aligned}
&r \cdot opt - (a_0 + a_1 + b_0)\\
&\geq r(a_1 + \frac{x_0a_0 + x_1b_0}{x_0}) - a_0 - a_1 - b_0\\
&= (r-1)a'_0 + (r\frac{x_1}{x_0} - 1)b_0 + (r-1)a'_1\\
&\geq a'_0[(r-1) + (r\frac{x_1}{x_0} - 1)\cdot\frac{b_0 + a'_1}{a'_0}]\\
&= a'_0r[1 - \frac{1}{r} + (\frac{x_1}{x_0} - \frac{1}{r})\cdot Q]\\
&= a'_0r[1 - (1 - \frac{x_1(x_0-x_1)}{x_0^2})\\
&+ (\frac{x_1}{x_0} - (1 - \frac{x_1(x_0-x_1)}{x_0^2}))\cdot\frac{x_1}{x_0-x_1}]\\
&= a'_0r[\frac{x_1(x_0-x_1)}{x_0^2}\\
&- \frac{(x_0-x_1)^2}{x_0^2}\cdot\frac{x_1}{x_0-x_1}]\\
&= a'_0r[\frac{x_1(x_0-x_1)}{x_0^2} - \frac{x_1(x_0-x_1)}{x_0^2}]\\
&= 0
\end{aligned}$$

This completes the proof of Case 1.

Case 2. $b_1 > 0$ **and** $a_0x_0 \geq b_1x_1$**:** Observe that by (c) $b_0 = 0$ and by (1) $opt \geq b_1 + a_1 + \frac{a_1x_1 + a_0x_0 - (b_1+a_1)x_1}{x_0}$. Note that $r - 1 - rx_1/x_0 < 0$, as $1 - 1/r = x_1(x_0 - x_1)/x_0^2 < x_1/x_0$. This implies the following (by (f) and $a_0x_0 \geq b_1x_1$):

$$\begin{aligned}
&r \cdot opt - (a_0 + a_1 + b_1)\\
&\geq r(b_1 + a_1 + a_0 - \frac{x_1b_1}{x_0}) - a_0 - a_1 - b_1\\
&= (r-1)(a_0 + a_1) + b_1(r - r\frac{x_1}{x_0} - 1)\\
&\geq (r-1)(a_0 + a_1) + \frac{a_0x_0}{x_1}\cdot(r - r\frac{x_1}{x_0} - 1)\\
&= a_1(r-1) + a_0(\frac{(r-1)x_0}{x_1} - 1)\\
&\geq 0
\end{aligned}$$

Case 3. $b_1 > 0$ **and** $a_0x_0 < b_1x_1$**:** By (1), $b_1 + a_1 = opt$, which implies the following (by (f) and $a_0x_0 < b_1x_1$):

$$\begin{aligned}
&r \cdot opt - (a_0 + a_1 + b_1)\\
&= (r-1)a_1 + (r-1)b_1 - a_0\\
&\geq (r-1)a_1 + (r-1)a_0\frac{x_0}{x_1} - a_0\\
&= a_1(r-1) + a_0(\frac{(r-1)x_0}{x_1} - 1)\\
&\geq 0
\end{aligned}$$

Case 4. $b_0 = 0$ **and** $b_1 = 0$**:** Observe that our solution $a_0 + a_1$ is optimal in this case, so we are done.

This completes the proof.

The above two lemmas immediately imply the following:

Theorem 3 *If $\alpha > 1/2$, then $R^{\infty}_{\{\alpha,\beta\}} = \frac{x_0^2}{x_0^2 - x_1(x_0-x_1)}$.*

Corollary 4 *[6] We have $R^{\infty}_{\{\alpha,\beta\}} = \frac{4}{3}$ for $\beta = \frac{1}{2} - \epsilon$ and $\alpha = \frac{1}{2} + \epsilon$ for any fixed nonnegative $\epsilon < \frac{1}{6}$.*

PROOF. Observe that $x_0 = 2$ and $x_1 = 1$, and apply the formula in Theorem 3.

3. When $\alpha \leq \frac{1}{2}$

Lemma 5 *If $1/3 < \alpha \leq 1/2$ and $x_2 + x_0 \geq 2x_1$, then $R^{\infty}_{\{\alpha,\beta\}} \leq \frac{x_0^2}{x_0^2 - x_2(x_0-x_2)}$.*

PROOF. When $x_2 + x_0 \geq 2x_1$ there exists an optimal solution with the following properties. Almost all bins contain either two α-items or no α-items, as if two bins contain one α-item each, then they can be rearranged so that one bin contains two α-items and the other contains no α-item.

We now use the algorithm given in Lemma 2, with item sizes β and 2α, by always placing either zero or two α-items in a bin (except possibly one bin). By the comment above on the optimal solution we get the desired bound from Lemma 2.

Lemma 6 *If $1/3 < \alpha \leq 1/2$ and $x_2 + x_0 < 2x_1$, then $R^{\infty}_{\{\alpha,\beta\}} \leq \frac{4}{3}$.*

PROOF. Before we describe the desired algorithm we prove a few claims, where $k = \lfloor \alpha/\beta \rfloor$.

Claim A. $x_1 = x_2 + k + 1$ **and** $x_0 = x_2 + 2k + 1 = 2x_1 - x_2 - 1$**.**

Note that $1-\alpha-(k+1)\beta \leq 1-2\alpha \leq 1-\alpha-k\beta$. So since exactly x_2 β-items will fit in a space of $1-2\alpha$, we will be able to fit at least x_2+k β-items in $1-\alpha$, but not more than x_2+k+1 β-items. Therefore $x_1 = x_2+k+i_1$, where $i_1 \in \{0,1\}$. Analogously $x_0 = x_1 + k + i_2$, where $i_2 \in \{0,1\}$. However $x_2 + x_0 < 2x_1$ implies that $x_2 + (x_1 + k + i_2) < x_1 + (x_2 + k + i_1)$, which in turn implies that $i_2 < i_1$. Therefore $i_1 = 1$ and $i_2 = 0$, which proves the claim.

Claim B. $\frac{x_1}{x_0} \leq \frac{3}{4}$ **and** $\frac{x_2}{x_0} < \frac{1}{3}$**.**

As $\alpha \geq \beta$, we observe that $k \geq 1$. Furthermore $k \geq x_2$, since otherwise $1-2\alpha \geq \beta x_2 \geq \beta(k+1) > \alpha$, a contradiction against $\alpha > 1/3$. Since $x_2 + 1 + (4k + 3x_2 + 3) \leq k + k + (4k + 3x_2 + 3)$, we get the following:

$$4x_1 = 4(k + x_2 + 1) \leq 3(2k + x_2 + 1) = 3x_0$$

This proves the first part of the claim. The second part follows from the fact that $3x_2 \leq 2k + x_2 = x_0 - 1 < x_0$. This completes the proof of claim B.

Now consider the algorithm that greedily places all items in bins, without ever putting an α-item and a β-item in the same bin. All bins, except at most two, will either contain x_0 β-items or two α-items. Assume that our algorithm produces a bins containing two α-items and b bins containing x_0 β-items. Note that there exists an optimal solution where either there is no bin containing two α-items or no bin containing zero α-items, as a bin with two α-items and a bin with zero α-items can be rearranged so that we get two bins each with one α-item and $x_2 + x_0 \leq 2x_1$. We aim to show $(a + b)/opt \leq 4/3$. The following three cases exhaust all possibilities.

Case 1. $bx_0 \geq 2x_1 a$**:** The optimal solution in this case must contain $2a$ bins each with one α-item (and x_1 β-items) and a further $\lceil \frac{bx_0 - 2ax_1}{x_0} \rceil$ bins containing no α-items. As we are considering the optimal asymptotic performance ratio, we may assume that the optimal solution uses exactly $opt = 2a + \frac{bx_0 - 2ax_1}{x_0}$ bins. By Claim B and $b \geq 2x_1 a/x_0$, this implies the following:

$$\begin{aligned}
opt &= 2a + b - 2a\frac{x_1}{x_0} - b/5 + b/5 \\
&\geq \frac{4}{5}b + 2a - 2a\frac{x_1}{x_0} + \frac{2x_1 a}{5x_0} \\
&= \frac{4}{5}b + a\left(2 - \frac{8x_1}{5x_0}\right) \\
&\geq \frac{4}{5}b + a\left(2 - \frac{8 \cdot 3}{5 \cdot 4}\right) = \frac{4}{5}b + \frac{4}{5}a
\end{aligned}$$

The above implies $(a + b)/opt \leq \frac{5}{4} \leq \frac{4}{3}$, which completes the proof of Case 1.

Case 2. $bx_0 < 2x_1 a$ **and** $bx_0 \geq x_2 a$**:** The optimal solution in this case must contain either one or two α-items in each bin. Assume that c bins contain exactly two α-items in an optimal solution. Note that $2c + (opt - c) = 2a$ and $x_2 c + x_1(opt - c) = bx_0$. By inserting $c = 2a - opt$ (from the first equation) into the second equation, we get $x_2(2a - opt) + x_1(2opt - 2a) = bx_0$, and hence $opt(2x_1 - x_2) = bx_0 + 2ax_1 - 2ax_2$. It follows, using $b \geq a\frac{x_2}{x_0}$, $x_0 + 1 = 2x_1 - x_2$ and $\frac{x_2}{x_0} < \frac{1}{3}$ (by Claim B) that

$$\begin{aligned} opt &= \frac{bx_0 + 2a(x_1 - x_2)}{2x_1 - x_2} \\ &= \frac{3}{4}b + a\frac{2x_1 - 2x_2}{x_0 + 1} + b(\frac{x_0}{x_0+1} - \frac{3}{4}) \\ &\geq \frac{3}{4}b + a\frac{x_0 - x_2 + 1}{x_0 + 1} + a\frac{x_2}{x_0}(\frac{x_0}{x_0+1} - \frac{3}{4}) \\ &= \frac{3}{4}b + a\frac{x_0 - x_2 + 1 + x_2}{x_0 + 1} - a\frac{3x_2}{4x_0} \\ &\geq \frac{3}{4}b + a - a\frac{1}{4} = \frac{3}{4}(b + a) \end{aligned}$$

This completes the proof of Case 2.

Case 3. $bx_0 < x_2 a$**:** In this case the optimal solution is $opt = a \geq \frac{3}{4}a + \frac{1}{4} \cdot \frac{bx_0}{x_2} \geq \frac{3}{4}(a+b)$, which completes the proof.

Lemma 7 *If* $\alpha \leq 1/3$*, then* $R^{\infty}_{\{\alpha,\beta\}} \leq 4/3$.

PROOF. We simply fill every bin greedily, without placing items of sizes α and β in the same bin. Note that almost all bins containing α-items (β-items) do not fit an additional item of size α (β).

Now consider a bin containing only α-items, which does not fit an additional item of size α. Let z denote the space left in the bin, and note that $z < \alpha$ and $z \leq 1 - 3\alpha$. Thus, $z \leq 1 - 3\alpha < 1 - 3z$. This implies $z < 1/4$. Analogously for every bin that contains only β-items and does not fit an additional item of size β, the space left in the bin is at most $1/4$. As all bins except possibly two are at least 75% full, we get the desired asymptotic performance ratio.

Lemma 8 *If* $0 < \beta \leq \alpha \leq 1$*, then* $R^{\infty}_{\{\alpha,\beta\}} \leq 4/3$.

PROOF. If $\alpha > 1/2$, then by Lemma 2 and $x_1(x_0 - x_1) \leq (\frac{x_0}{2})^2$ (this follows from $(x_0 - 2x_1)^2 \geq 0$) we have

$$R^{\infty}_{\{\alpha,\beta\}} \leq \frac{x_0^2}{x_0^2 - x_1(x_0 - x_1)} \leq \frac{x_0^2}{x_0^2 - (\frac{x_0}{2})^2} = \frac{1}{1 - \frac{1}{4}} = \frac{4}{3}$$

If $1/3 < \alpha \leq 1/2$ and $x_2 + x_0 \geq 2x_1$, then by Lemma 5 we have

$$R^{\infty}_{\{\alpha,\beta\}} \leq \frac{x_0^2}{x_0^2 - x_2(x_0 - x_2)}.$$

Similarly to the previous argument, we can now prove that $R^{\infty}_{\{\alpha,\beta\}} \leq \frac{4}{3}$.

If $1/3 < \alpha \leq 1/2$ and $x_2 + x_0 < 2x_1$, then Lemma 6 implies the desired result. We are now done by Lemma 7.

Lemmas 1 and 8 imply immediately the following:

Theorem 9 *We have* $\max\{R^{\infty}_{\{\alpha,\beta\}} : \ \alpha, \beta \in (0,1]\} = 4/3$.

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References

[1] D.J. Brown, A lower bound for on-line one-dimensional bin packing algorithms. Tech. report R-864, Coordinated Science Laboratory, Urbana, IL, 1979.

[2] B. Chandra, Does randomization help in on-line bin packing? Inform. Process. Lett. 43 (1992) 15-19.

[3] E.G. Coffman, Jr., M.R. Garey, and D.S. Johnson, Approximation Algorithms for Bin Packing: A Survey. Approximation Algorithms for NP-Hard Problems, D. Hochbaum (editor), PWS Publishing, Boston (1996), 46-93.

[4] E. G. Coffman, Jr., G. Galambos, S. Martello and D. Vigo, Bin Packing Approximation Algorithms: Combinatorial Analysis. Handbook of Combinatorial Optimization, Supplement Volume A, D.-Z. Du and P.M. Pardalos (editors), Kluwer (1999) 151-208.

[5] J. Csirik and G. Woeginger, On-line packing and covering problems. On-line Algorithms - the State of the Art, Lect. Notes In Comput. Sci. 1442, A. Fiat and G. Woeginger (editors), Springer, New York (1998), 147-177.

[6] U. Faigle, W. Kern and G. Turan, On the performance of on-line algorithms for partition problems. Acta Cybernetica 9 (1989), 107-119.

[7] M. R. Garey, R.L. Graham and J.D. Ullman, Worst-case analysis of memory allocation algorithms. In Proc. 4th ACM Symp. Theory Comput. (1972) ACM, 143-150.

[8] F.M. Liang, A lower bound for online bin packing. Inform. Process. Lett. 10 (1980) 76-79.

[9] S. Seiden, On the online bin packing problem. J. ACM 49 (2002) 640-671.

[10] A. van Vliet, An improved lower bound for online bin packing algorithms. Inform. Process. Lett. 43 (1992) 277-284.

[11] J.D. Ullman, The performance of a memory allocation algorithm. Tech. Report 100, Princeton Univ., Princeton, NJ, 1971.

[12] A.C.C. Yao, New algorithms for bin packing. J. ACM 27 (1980) 207-227.

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