



# Robust Selling Times in Adaptive Portfolio Management

Michael Dziecichowicz

Department of Industrial and Systems Engineering, Lehigh University, 200 W Packer Ave, Bethlehem PA 18015, USA

Aurélie Thiele

P.C. Rossin Assistant Professor, Department of Industrial and Systems Engineering, Lehigh University, 200 W Packer Ave, Bethlehem PA 18015, USA

## Abstract

*Traditional techniques in portfolio management rely on the precise knowledge of the underlying probability distributions; in practice, however, such information is difficult to obtain because multiple factors affect stock prices on a daily basis and unexpected events might affect the price dynamics. To address this issue, we propose an approach to dynamic portfolio management based on the sequential update of stock price forecasts in a robust optimization setting, where the updating process is driven by the historical observations. Forecasts are updated using only the most recent data when the stock price differs significantly from predictions. In this work, we present a robust framework to optimal selling time theory. We introduce a wait-to-decide period, and allow actual price movements to drive the best decision in response to a bad investment. Numerical results illustrate our strategy, which requires less frequent updating of the problem parameters than in the traditional approach while exhibiting promising performance.*

*Key words:* confidence intervals; trigger thresholds; portfolio management.

## 1. Introduction

The foundation of modern portfolio management was developed in the 1950s when Markowitz formulated a single-period portfolio selection problem where the investor seeks to maximize his utility by investing in assets with random returns (Markowitz 1952). If the returns follow a jointly normal distribution and the investor's utility is quadratic, the problem can be reformulated as a mean-variance problem that maximizes the expected return while constraining the portfolio variance. Sharpe (1964), Lintner (1965), and Mossin (1966) extended this mean-variance model when they independently developed the capital asset pricing model (CAPM), which laid out a set of assumptions characterizing investors' behavior. By simplifying how an investor acts and reacts to market movements, Treynor (1966), Sharpe (1966), and Jensen (1969) recognized

*Email:* Michael Dziecichowicz [mjd404@lehigh.edu], Aurélie Thiele [aurelie.thiele@lehigh.edu].

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the CAPM to be a practical method to compare the performance of all active portfolio managers.

Recently, research has strayed away from the underlying assumptions of the Markowitz model: asset returns are not necessarily Normally distributed, and an investor's utility is not necessarily quadratic. It is very difficult for an investor to articulate his utility, and estimating the distribution of asset returns is a daunting task. In addition, optimal portfolios are often sensitive to estimated parameters, in that the optimal allocation solved using inaccurate parameters might vary significantly from the optimal allocation using true values (Chopra and Ziemba 1993). This calls for the development of robust models in portfolio optimization that rely on minimal parameter estimation. Gülpınar and Rustem (2007a,2007b) and Rustem et. al. (2000) extend the classical mean-variance portfolio problem using a robust min-max approach to address imprecise return forecasts and risk estimation. They characterize optimal decisions from a worst-case scenario perspective, and suggest various approaches to handling future uncertainty while guaranteeing noninferior performance. Our work, instead, builds on such stochastic models by reacting to new observations quickly and allowing real price movements to direct decisions in all scenarios as

time progresses.

Furthermore, the CAPM model presents several other issues. Under the assumption that markets are efficient, it is difficult for managers to earn profits after accounting for transaction costs and research fees. Also, a second assumption that an investor pays no transaction costs or incurs federal and state tax on gains leads to an unrealistic portfolio revision policy; this assumption suggests forecasting and updating as often as possible. Bertsimas and Pachamanova (2008) provide a detailed study on various robust portfolio optimization approaches, and highlight the computational advantage of a simple underlying model when handling complex additional requirements, e.g., transaction costs.

As an alternative to risk-adjusted returns, practitioners often wish to decide which decisions result in superior performance, which depends on their ability to invest in the best-performing securities at the best time relative to the investor's objectives. Investors face both general and detailed choices when selecting a portfolio. General assessment of timing and selection ability include deciding to invest in equities as opposed to fixed-income securities when the stock is performing well. A significant body of literature has been dedicated to assessing peaks, bottoms, trends, patterns and other factors affecting a stock's price movements to determine future values. (Examples of often-used indicators include Williams %R, Commodity Channel Index rating, Money Flow Index, and Average Directional Index ratings and moving average crossovers.) Chance and Hemler (2001) study real market data of professional timers and find evidence of timing capability. The authors use robust regression techniques to validate their findings, but noted that the professionals who traded most frequently performed best. When accounting for taxes and fees timing performance diminished significantly, but not completely.

Specific assessment focuses on more detailed allocation choices, such as choosing stocks in a better-performing industry or the relatively better-performing stocks in a given industry. Many studies have been conducted to that effect, noting that skilled active money managers can outperform the average forecasts built into market prices used in passive strategies; for instance, the reader is referred to Klaas et. al. (2001) for a discussion on the economic significance of investing in active funds, even in cases where the investor is skeptical about the manager's skill level. Huang et. al. (2006) present a robust Conditional Value-at-Risk approach to portfolio selection with uncertain exit time. They make

allocation decisions accounting for exogenous and endogenous exit probabilities, and solve the resulting optimization model over an interval of exit probabilities to protect against downside risk. Approximating exit probabilities at future time steps differs markedly from allowing real price movements to drive the portfolio allocation, which we do here.

In contrast, we propose an approach to dynamic portfolio management based on the sequential update of stock price forecasts in a robust optimization setting, where the updating process is driven by historical observations and the information that recent price changes reveal. Forecasts are updated using only the most recent data when the stock price differs significantly and consistently from predictions. In other words, this work focuses on determining a fund manager's next-best-step once he realizes he might have made a mistake, while avoiding to sell an asset too soon in case poor early performance is not representative of the true stock price dynamics. Karatzas and Wang (2000) study optimal stopping times with discretionary stopping, whereby the decision-maker can freely stop before or at a prespecified final time in order to maximize the expected utility of his portfolio wealth or consumption. This approach relies on finding an optimal family of strategies based on a calculated optimal return function. In contrast, performance in our model is assessed using a manager's regret, defined as the difference between a scenario outcome for the manager and a specified benchmark strategy, e.g., the stock price at the end of the time horizon. Regret provides a computationally tractable downside risk measure without requiring any assumption on the investor's utility function or the distribution of the returns.

Most works in behavioral finance define regret to be "a psychological reaction to making a wrong decision where wrong is determined by actual outcomes rather than in relation to the information available at the time the decision was made" (Dembo and Freeman 1998). In this sense, behavioral theorists have used the notion of regret to shed light on why investors, managers, and analysts have a tendency to invest and make decisions similar to each other, a phenomenon often referred to as "piggy-backing". Individuals, and fund managers, tend to feel less regret when they lose money along with others, as they are less subject to intense scrutiny. Our notion of regret, which focuses on identifying early that a mistake has been made to avoid compounding its effect, provides a unique approach to market timing opportunities and allows managers to take unique, original

investment decisions while limiting their downside risk.

Further motivation for this study stems from traditional decision theory, in which a stop-loss rule determines whether to continue or stop a process based on the present position and past events. In this work, we refer to the non-robust approach as the “traditional” approach. We add robustness into the decision-making framework by introducing a waiting period prior to the decision to sell risky assets once their value falls below a threshold. Thus, the manager faces two separate investment decisions. First, how often should the manager update the parameters and how should the most recent information be incorporated into his decision? Second, if the manager receives a signal to consider adjusting his portfolio allocation, how soon after the signal should the adjustment occur, i.e., when should the manager stop hoping for a correction and cut his losses? Xia (2001) addresses similar issues in the context of dynamic asset allocation, but focuses on how to update predictor parameters at each time period, and not on when model parameters should be updated.

This notion of robustness attempts to value obtaining additional information relative to the amount risked in doing so. What we hope to achieve is a framework for decision-making that decreases the chance for the manager to realize he has made a mistake when he reaches the end of the time horizon, and the size of that mistake (measured by the amount of money lost). Performance is evaluated by comparing the approach with traditional strategies, such as selling right away or holding the stock until the end. We agree that the key to successful management is in the search for accurate and superior information (Bodie et al. 2005). While, in practice, managers are often quick to sell when stock prices fall, our numerical results show that this is not always the optimal strategy. The results also suggest that such frequent, e.g., daily, assessments may lead to poor decision-making: if the decision-maker realizes soon after selling that he should not have sold, the decision was not robust.

Our model builds upon the Lognormal distribution of stock prices, which has been shown to provide a reasonably good starting point to fit real data (see Hull (2002) and the references at the end of Chapter 12 for a discussion of the model and its limitations); our goal is to strengthen this framework by introducing flexibility (adaptability) in the decision-making process. Specifically, our work seeks to improve on existing portfolio management techniques by incorporating newly observed data into probabilistic decision thresholds. We

further investigate our ability to reduce unnecessary updating in order to save portfolio managers time, computational effort, and fees in practice. Results indicate that the proposed approach plays a significant role in helping to recognize when a bad investment has been made and when an asset should be sold. We demonstrate that our approach requires less frequent updating of the problem parameters than the traditional approach while preserving performance. To the best of our knowledge, this is the first time that an adaptive forecasting approach has been proposed to the optimal stopping (selling) time problem in portfolio management.

The remainder of this paper is organized as follows. Section 2. presents the model, numerical simulation, results, and analysis for one risky asset. The model is extended to multiple risky assets in Section 3.. Section 4. contains concluding remarks.

## 2. The Model

### 2.1. The Binomial Approximation to Stock Dynamics

We consider an asset pool of  $N$  risky assets in a discrete-time, finite-horizon setting. There exist no transaction costs for selling or buying assets and each risky asset  $n \in N$  is in supply of  $S_n$  shares. The fund manager is interested in buying shares of assets that return maximum profit to his clientele, while adhering to their loss restrictions. He has an initial wealth of  $W_0$  to allocate between the risky assets and a riskless asset earning an exogenous rate  $r$ , and must hold  $M < N$  assets at all times until  $T$ , at which point he liquidates the entire portfolio. He tracks each asset, and buys and sells assets that have fallen below or risen above set probability thresholds. When he decides to sell assets, he must reinvest the liquidated funds in either a risky or riskless asset immediately.

A vast amount of literature supports the conjecture that stock price movements follow a Lognormal distribution, which can be approximated by a binomial random walk. (The reader is referred to Ison (1996) and Treynor (1966) for a succinct presentation of practical insight into the binomial approximation of stock price movements.) Net gain realized by holding stock  $n$  is described as a random walk  $Y_{tn} = \prod_{s=1}^t X_{sn}$  at time  $t$  with all  $X$ 's obeying the distribution of the random variable  $X_n$ :

$$X_n = \begin{cases} u_n, & \text{w.p. } p_n, \\ d_n, & \text{w.p. } q_n = 1 - p_n, \end{cases} \quad (1)$$

where  $u_n, d_n$  are known constants with  $d_n < 1$  and  $1 + r < u_n$  (so that the choice between the risky and the riskless assets is non-obvious).

A manager’s expected profit from stock  $n$  is described by the estimated probability  $\hat{p}_{tn}$  the stock increases in price each time period. Instantaneous mean  $\mu_{tn}$  and standard deviation  $\sigma_{tn}$  of the random profit are:

$$\begin{aligned} \mu_{tn} &= \ln u_n \cdot (2\hat{p}_{tn} - 1) \\ \sigma_{tn} &= \sqrt{\hat{p}_{tn}(\ln u_n - \mu_{tn})^2 + (1 - \hat{p}_{tn})(\ln u_n + \mu_{tn})^2} \end{aligned} \quad (2)$$

The manager initially decides to invest in a stock if he believes the stock will have greater return than the riskless asset over the investment horizon  $[0, T]$ . Using profit parameters  $(\mu_{tn}, \sigma_{tn})$ , we find the probability that the stock price will rise above or fall below a specified level at each time  $t$ , for instance, the probability that the price at expiration is above or below the price at which he bought an asset. The decision to sell at time  $t$  a stock bought at time  $t_0$  is made when:

$$\Pr(Y_{Tn} \leq S_{t_0n}) \geq \epsilon_s,$$

where  $S_{t_0n}$  is the price of stock  $n$  at time  $t_0$  and  $\epsilon_s$  specifies the probability of loss. Using that  $Y_{Tn} = S_{tn} \prod_{i=t+1}^T X_{in}$ , this can be written as:

$$\Pr\left(S_{tn} \prod_{i=t+1}^T X_{in} \leq S_{t_0n}\right) \geq \epsilon_s. \quad (4)$$

Taking the logarithm and rearranging terms in Equation (4), we construct confidence intervals on a manager’s decision to sell at time  $t_{(s)}$  depending on time left until expiration  $T - t$ . We have:

$$S_{t_{(s)}n} \leq S_{t_0n} e^{-[\mu_{tn}(T-t) + \phi^{-1}(\epsilon_s)\sigma_{tn}\sqrt{T-t}]}. \quad (5)$$

where  $\phi^{-1}$  is the inverse of the t-distribution with  $t - t_0 - 1$  degrees of freedom. A manager sells assets when they fall below the threshold (5); this will be referred to as a trigger threshold for the remainder of this paper.

### 2.2. Robust Selling Time Approach

In practice, well-performing stocks often exhibit poor performance over a few time periods, and should be kept in the fund’s portfolio nonetheless. Because of fast-changing market conditions, driven by changes in leadership, quarterly earnings announcements, competitors’ moves and global market conditions, it is difficult to assess whether a stock will perform well in the future

based on historical data. Managers are particularly keen on avoiding selling stocks too early, as this mistake is much more obvious to their clients than that of not investing in a given stock at all. Our purpose here is to make the manager’s decision more robust by defining the selling time above to be the start of a stopping process. The manager considers selling the stock when the probability of the stock getting worse exceeds threshold (5); at this time, he questions the validity of the estimated parameters but updates the parameters accordingly rather than selling immediately. Going forward, the manager selects one of several possible options, depending on stock behavior.

The manager chooses to sell the stock if losses incurred are too severe (the stock crosses an “immediate sell” threshold when the stock is down) or if the price has not recovered enough at the end of a waiting period. The “immediate sell (down)” threshold is identical to Equation (5), except that the probability of loss at expiration  $\epsilon_{isd}$  is higher.

$$S_{t_{(isd)}n} \leq S_{tn} e^{-[\mu_{tn}(T-t) + \phi^{-1}(\epsilon_{isd})\sigma_{tn}\sqrt{T-t}]}. \quad (6)$$

The decision to sell may also be the result of a sharp increase in price. In this case, the manager believes that the price has peaked and he wishes to capture abnormal gains. We model this event using a similar threshold where  $\epsilon_{isu}$  is the level of confidence the manager has that the price has peaked and will not continue to rise; this addresses the mean-reverting tendency exhibited by stocks in practice.

Second, if the price does recover enough in the stopping period to justify holding the stock longer, a manager can decide not to sell at this time. In this instance, the manager updates his profit estimation parameters and recomputes his decision thresholds. The stopping process terminates and will not begin again unless the probability of loss again exceeds a threshold reflecting updated information. The recovery threshold is identical to (5), except the probability of loss at expiration  $\epsilon_r$ , has decreased:

$$S_{t_{(r)}n} \geq S_{tn} e^{-[\mu_{tn}(T-t) + \phi^{-1}(\epsilon_r)\sigma_{tn}\sqrt{T-t}]}. \quad (7)$$

If the stock neither spikes or recovers, the manager must decide whether to sell the stock or wait to see if the stock will recover. By waiting, the manager obtains more information on the price process but might incur even bigger losses; an important question arises then regarding the length of the waiting period. In or-

der to determine the appropriate length, we consider the expected hitting time  $\tau$  of either level  $A$  or  $-B$  (with  $A, B > 0$ ) for the biased random walk. Recall that:

$$E(\tau|S_0 = 0) = \frac{B}{q-p} - \frac{A+B}{q-p} \frac{1 - \left(\frac{q}{p}\right)^B}{1 - \left(\frac{q}{p}\right)^{A+B}},$$

where  $q = 1 - p$ . Our model cannot directly apply this formula because our thresholds are not constant, and it is not possible to find a closed-form solution. To derive an approximate expected hitting time, we rely on conditional hitting probabilities at each time period from the beginning of the stopping process to expiration. We find that they require significant computational effort, and the decision thresholds change very little at times far from time  $T$ , such that nearly identical results are obtained by simply fitting a horizontal line to the non-linear thresholds. At times near  $T$ , however, the expected hitting time increases rapidly because of the way the bounds approach the price level  $S_{0n}$  asymptotically. Thus, we set our loss allowance at 2% to solve for a one-sided expected hitting time, which we use in the simulations presented below. We know that in expectation,

$$E[S_n] = \ln u_n \cdot (2\hat{p}_{tn}^* - 1), \quad (8)$$

where  $\hat{p}_{tn}^*$  represents the sample probability at the time the stock price hits the threshold triggering the waiting period. We then solve for  $\hat{p}_{tn}^*$  using:

$$\hat{p}_{tn}^* = \frac{1}{2} \left[ \frac{\ln S_{t(s)n}}{t \cdot \ln u_n} + 1 \right].$$

Substituting  $\hat{p}_{tn}^*$  into Equation (8) and using Wald's equation, we find the value for the expected hitting time as:

$$E[\tau] = -\frac{x}{E[S_n]}. \quad (9)$$

where  $x$  (in %) represents the maximum loss the manager is willing to accept going forward.

The solution to Equation (9) provides a minimum for the length of the stopping period, given that the process does not fall below its immediate-sell threshold or recover enough to end the triggered process. The actual length of the stopping period is determined by the value of the stock price at the expected hitting time, conditioned on the price rising in the next time period. In other words, if a stock price increase in the next time period results in the price exceeding the "sell" trigger threshold, the length of the stopping period will increase

by one period. If the price moves higher, the stopping process horizon will be extended until the manager finds the stock has recovered. If the price falls lower and more than  $E[\tau]$  periods have passed, the manager will sell.

### 2.3. Framework and Calibration

Suppose initial wealth of each manager is  $W_0 = 1$ . There exists one risky and one riskless asset for each manager to choose from in the selection of his portfolio, and he must invest his entire wealth at all time periods. A manager selects a stock to purchase shares of at time  $t_0 = 1$  based on expected performance. We model this decision by assuming  $\hat{p} > 0.5$  at  $t_0$ . We assume that the manager cannot decide to sell a stock for a minimum of 2 business weeks after the purchase, as he will be reluctant to admit he has made a mistake. If the price of the stock falls below the loss threshold, the manager must sell immediately and invest in a riskless asset until time  $T = 60$ . If a manager decides to sell the risky asset, he must invest the entire proceeds from selling stock into the riskfree asset at  $r = 3\%$  annually. A manager can decide to sell at any time period, and each time period represents one trading day. As mentioned above, there is no transaction fee to sell a risky asset or to invest in a risk-free asset.

As noted in Equations (2) and (3), the parameter  $\hat{p}$  dictates our decision-making process; thus, our objective is to update this parameter whenever there is reason to believe it is not accurate. We seek to differentiate between the cases where a manager has invested in a bad stock and when he has incorrect subjective probabilities. An example of investing in a bad stock is investing in a company taking on a new CEO, who makes poor business choices and causes the value of the stock to drop; if a manager believes the CEO is good for the company, he will invest in the stock, but must quickly realize that his investment choice was a mistake. On the other hand, suppose a manager invests in a soaring stock which has produced strong gains over the past few weeks. If the stock price falls due to some temporary isolated market noise soon after the manager has purchased shares, the manager must realize that the strength of the company has not changed. Therefore, we construct our model so that we can closely monitor the implications of assuming a given value of  $\hat{p}$  relative to its closeness to the actual probability of the stock movement  $p$ .

We generate 5000 stock prices assuming the real probability  $p$  of the stock price is known using MATLAB. We select uniformly distributed values for  $p$  over

[0.48, 0.505], to focus on the range where the decision is not clear-cut. For each value of  $p$ , we find values for  $u$  and  $d$  in Equation (1) by setting  $d = \frac{1}{u}$  and solving the equation in  $u$ :

$$p = \frac{a - \frac{1}{u}}{u - \frac{1}{u}},$$

or, equivalently:

$$u^2p - au + 1 - p = 0,$$

where  $a = e^{rdt}$ .

In Figure 1, we see that for values of  $p$  below 0.48,  $u$  takes on unreasonable values to represent daily increments. Also, for values greater than 0.505,  $u$  flattens out such that our model does not reflect any other performance change than for  $p = 0.505$ .

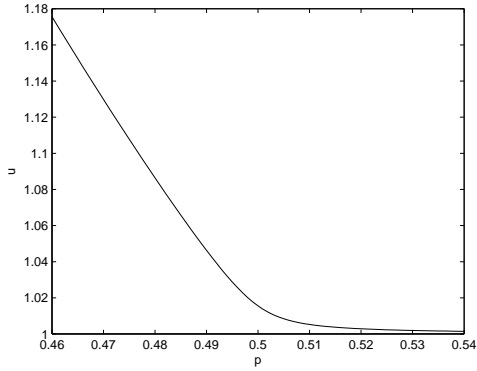


Fig. 1. Solving for  $u$  over  $p$

Also, for each  $p$ , we run our simulation over an interval of  $\hat{p} = [0.5, 0.7]$  in increments of 0.05. (This interval is larger than the interval for  $p$  because managers only estimate over a short time horizon. Thus, the estimated parameters will change rapidly as time progresses.) The interval was chosen as such to ensure that the manager will decide to purchase the risky asset at  $t_0$ , while ensuring that managers do not assume unrealistically high gains.

The only remaining parameters that need to be defined are the values that  $\epsilon$  have been set to in creating decision thresholds. We analyze the performance of the approach for various values of  $\epsilon_s$ , and choose values of  $\epsilon_r$  and  $\epsilon_{isd}$  so that the feasible price range for the robust selling period bounds  $S_{t(r)}$  and  $S_{t(isd)}$  is approximately  $2\sigma_t$  in length each time thresholds are updated. For example, when  $\hat{p} = 0.6$ ,  $\epsilon_s = 0.5$  results in best overall

returns, and thus we set  $\epsilon_r = 0.31$  and  $\epsilon_{isd} = 0.77$ .

In order to assess the performance of the decision mechanism, we compare the mean outcome for a fund manager who acts according to the robust mechanism detailed above to that of a manager who opts for a traditional strategy. A (sell) mistake is defined as a decision which results in the manager receiving less than he would have by holding onto the stock until time  $T$ , and the difference is the cost incurred. A correct (sell) decision results in the manager earning more than he would have by holding onto the asset until  $T$ , and the difference is the amount saved. Thus, the average comparative return  $\bar{S}$  is:

$$\bar{S} = \frac{\text{amount save} - \text{amount cost}}{\# \text{ of diverging decisions}}. \quad (10)$$

The results below suggest that our model also provides valuable insights into the value of gaining information before making a decision to sell.

## 2.4. Simulation Results

### 2.4.1. Overall Performance

The first simulation presented examines the flexibility of the robust model. For each value of  $\hat{p}$  in the range [0.5, 0.7], we implement the robust decision mechanism on a set of 5000 generated stock processes, with  $T = 60$  time periods. Recall that positive values of  $\bar{S}$  indicate that the robust approach outperforms the traditional one. Results from this simulation are as follows:

Table 1

Performance (across all $\hat{p}$ ).				
$p$	$\bar{S}^*$ (\$)	max	precision(%)	$\Delta t$
0.48	-0.00128	0.3063	3.83	7.20
0.485	-0.00212	0.2117	4.53	7.18
0.49	0.00013	0.2032	6.74	7.25
0.495	0.00091	0.1254	9.68	7.28
0.5	0.00051	0.0572	11.77	7.50
0.505	0.00027	0.0303	11.15	7.59

where:

- $\bar{S}^*$  is defined in Equation (10),
- **max** is the average difference between the maximum amount saved and the maximum cost in the traditional vs robust approaches.
- **precision** is the average percentage of avoided mistakes using the robust approach relative to the traditional approach,

- $\Delta t$  is the average number of extra time periods a manager using the robust approach invests in risky assets.

We find that our model performs worse than the traditional approach for the two lowest values in the range of  $p$ . This is due to the extreme volatility linked to low  $p$  values, or unrealistic magnitudes of  $u, d$ . Results are very encouraging if  $p$  is restricted to a smaller, more practical range  $[0.49, 0.505]$ . The following table presents results for the case where managers invest in a riskless asset after selling the risky asset:

Table 2

*Mean performance across all  $\hat{p} = [0.49, 0.505]$ .*

$p$	$S^*$	max	precision(%)	$\Delta t$
$mean_{all}$	0.00046	0.10403	9.84	7.41

The results show that the robust approach performs better than the traditional approach on average over this interval of  $p$ . The gap between the maximum amount saved and cost is tightened by \$0.10 on average. The manager using the robust model correctly identifies 9.84% more mistakes relative to the traditional model. Also, the robust approach will hold risky assets an average of 7.41 time periods longer than the traditional approach. While the increase in  $\bar{S}$  is not big, one must keep in mind that realize that this reflects only the improvement with only one stock and a manager who begins with \$1.

Looking back at Table 2, we find that our model performs best when  $p = 0.495$ . Relative to other values of  $p$ , the values of **max** and **precision** imply that the best performing managers are those who balance limiting the variation of their returns and increasing their precision. An overly conservative manager (low  $\hat{p}$ ) will greatly decrease the gap of max outcomes but will earn returns very similar to the traditional case. A manager who focuses on maximizing his precision must bear with very volatile outcomes, and take on extreme losses more often. In the next section, we run a different simulation to compare the performance of different fund managers; their differences are reflected in their different choices for  $\hat{p}$ .

#### 2.4.2. Heterogeneous Managers

For each value of  $\hat{p}$  in the range  $[0.5, 0.7]$ , we implement the robust decision mechanism on a set of 5000 generated stock processes with  $T = 60$  time periods. We then change the initial value of  $\hat{p}$  (which does not

affect the processes) and the respective values for  $\epsilon$ . Results from this simulation are as follows:

Table 3

*Heterogeneous Managers  $\hat{p} = [0.5, 0.7]$*

$\hat{p}$	$S^*$	max	precision(%)	$\Delta t$
0.5	0.000292	0.8798	14.65	10.54
0.55	0.000372	0.7445	9.51	7.79
0.6	0.000890	0.6459	8.20	7.12
0.65	0.000042	0.6160	6.53	6.22
0.7	0.000398	0.5089	8.63	5.28

where all parameters are defined as before. Notice that here we average across all values of  $p$  for each value of  $\hat{p}$ , whereas in the previous simulation we averaged across all values of  $\hat{p}$  for each value of  $p$ .

We find that the manager with  $\hat{p} = 0.6$  is the top performer with respect to  $\bar{S}$ . In line with the results presented in the previous section, this manager balances his limitation of taking an extreme loss, the desire to increase his precision, and his willingness to invest in risky assets longer than is recommended by traditional approaches. Managers who estimate their parameters too conservatively will exhibit the greatest precision relative to the traditional approach, but the increase in performance is small on average. These managers concern themselves with better behaved processes which have lower standard deviation, which explains the longer average waiting period and greater reduction in the variability of outcomes. Managers who take on too much risk, on the other hand, set their parameters such that recovery is harder and they cut their losses too soon. Also, they are more willing to accept a greater magnitude of loss, so the variability of their outcomes is relatively high.

We conclude that a manager who follows the robust approach does the best at differentiating between an investment mistake and a temporary downward price trend. We see that neither maximizing precision nor holding onto risky assets as long as possible is an optimal objective for investing.

#### 2.5. Example

Figure 2 illustrates an example of the decision process faced by a manager in one specific run. The solid line represents the stock price. Each dotted line represents a threshold defined above, and is labeled accordingly. Suppose at  $t = 0$ , the manager assumes a  $\hat{p}$  value of 0.6, and thus invests his entire wealth  $W_0$  in shares of

the stock. At time  $t = 38$ , the stock first falls below the trigger threshold  $S_{t_s}$ , which is represented as a circle on the plot. A manager following the traditional approach would sell his stock holding at that time period and reinvest his funds in a riskfree asset until  $T$ . It is obvious in this case that this “traditional” manager sells too soon. Had he given the process a chance to recover, he would have earned a significant profit.

A manager following the robust approach decides to begin a stopping process, updates the value of  $\hat{p}$  at  $t = 38$  and holds the stock instead in hopes of recovery. At this time,  $\hat{p} \approx .43$ . This value, along with the thresholds (6) and (7) from  $t = 38$  to  $t = T$ , are used to find the expected hitting time. In this run,  $E[\tau] = 9$  time periods, so that we wait until  $t = 47$ . The process behaves well in the waiting period, and it appears to be en route to recovery at  $t = 47$ , as it has risen above the trigger threshold. Thus, we allow the stopping period to be extended until either the process has recovered or until it decidedly loses momentum and is sold. The process crosses its recovery threshold (7) at  $t = 52$ , and all parameters are updated, and a new set of thresholds are constructed using this newly available information. At time  $T$  we find that the manager using the traditional approach has made a mistake, and ends with a 46% loss over the entire time interval. The manager following the robust approach has correctly identified that the decrease in price was not a mistake, and receives a 45% profit relative to the price at purchase.

### 3. Multiple Risky Assets

#### 3.1. The Approach

We now introduce multiple risky assets into the portfolio mix. The decision to reinvest liquidated funds in either risky or riskless assets is determined by maximizing the Sortino ratio of the portfolio mix. This downside risk measure, introduced by Sortino and Price (1994), replaces standard deviation with downside semi-standard deviation in Sharpe’s famous measure. Thus, the ratio is the expected rate of return in excess of the riskless asset, per unit of downside risk.

In the case where the manager is holding the riskless asset, he decides to swap back into some risky asset when the return to-date exceeds that of the riskless asset (recall that we assume no transaction costs), i.e., when:

$$\left[ \frac{S_{tn} - S_{1n}}{S_{1n}} \right]^{1/t} > r.$$

The manager is restricted from buying back into the same portfolio he just sold; furthermore, a manager will not immediately buy back into a portfolio that he has recently convinced himself and his clientele to be a bad investment strategy.

A last necessary addition to the multiple risky assets model is an aggregate portfolio downside threshold. If that aggregate threshold is crossed, the manager will liquidate the risky assets and allow the wealth to sit in a riskless investment until time  $T$ . The level of this threshold is determined using unrestricted simulations (that is, without the threshold) and setting it equal to a percentage of the maximum observed losses; we allow the percentage to depend on the size of the losses. For example, if a manager is holding a risky portfolio worth \$100 today, which in the past quarter hit a low of \$60 in aggregate value, he will only accept a maximum loss of say 40% of \$40, whereas a manager who is holding a risky portfolio worth \$100 today, which in the past quarter hit a low of \$94, will be willing to accept a maximum loss of say 80% of \$6.

#### 3.2. Problem Setup

##### 3.2.1. Questions

We now are faced with a broader set of questions to assess our strategy’s performance in the multiple asset case: Does the decision mechanism described produce higher returns than traditional stopping rules? Are results consistent with the findings above regarding the flexibility of our approach and heterogeneous managers? How volatile are the terminal values of the portfolio? How much variability exists in the maximum and minimum performances across all trials? Does maintaining a robust decision mechanism for the entire time interval compound the effects of a mistake relative to the traditional approach? Ultimately, has the robust model increased the managers’ ability to recognize and optimally react to mistakes?

##### 3.2.2. Framework

Suppose the initial wealth of each manager is  $W_0 = 100$ . There exists a pool of  $N = 10$  risky assets and one riskless asset to choose from in the selection of the portfolio, and each manager must invest his entire wealth at all time periods. A manager selects  $m = 5$  stocks to purchase at time  $t_0$ . A manager will not sell a stock for a minimum of 2 business weeks after purchase; if he decides to sell an asset he is holding, he must invest the entire proceeds from selling the asset into some other



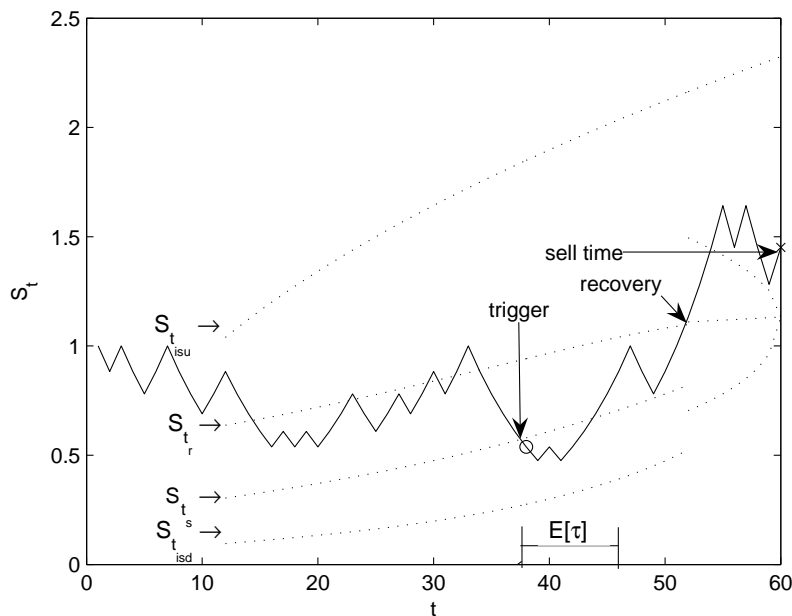


Fig. 2. Example.

investment not being held. A manager can decide to sell at any time period, and each time period represents one day or  $\frac{1}{252}$  of a year. Assume no withdrawals of funds by investors and no transaction fees to sell or buy an asset.

We compare the mean outcome for a fund manager who acts according to the robust mechanism detailed above and that for a manager who opts for a traditional strategy. In each run of each instance, we collect data across the same ranges for  $\hat{p}$  and  $p$  as in the single asset case. First, we track the difference between the final portfolio value and the value at the time the first mistake is realized. By doing so, we can assess which strategy recovers better from this first downfall in wealth, as both managers' portfolios have identical worth at this moment. This statistic is the principal performance measure. Second, the standard deviation of realized portfolio values is averaged over all values of  $p$  and  $\hat{p}$ . This indicates how volatile the aggregate portfolio values are from the time of the first mistake until time  $T$  for all runs. Next, a comparison of the maximum and minimum portfolio values provides a check to ensure our approach is in line with robust theory. Last, we keep count of the total number of decisions made for each value of  $p$  and  $\hat{p}$  in order to determine how much effort the robust strategy saves a manager. The swapping

mechanism between stocks was coded and the numerical experiments were performed using MATLAB.

### 3.3. Results

#### 3.3.1. Overall Performance

As in the single risky asset case, we first test the flexibility of the robust model. For each value of  $\hat{p}$  in the range  $[0.5 \ 0.7]$ , we implement the robust decision mechanism on a set of 2000 runs, each involving a pool of 10 risky assets and one riskless asset. A change in the value of  $\hat{p}$  warrants the generation of new stock processes, and requires a change in the respective values of  $\epsilon$ . For simplicity in the experiments, we assume that the  $\hat{p}$  values of the 5 stocks in the portfolio at the beginning of the time horizon are the same for each run. Results from this simulation are as follows:

- where:
- **AP** is the average performance difference,
  - $\sigma_{diff}$  is the difference between the average standard deviation of all realized portfolio values,
  - **min** is the average difference of minimum performance,
  - **max** is the average difference of maximum performance,
  - **# dec.** is the average difference of total number of

Table 4

*Performance (across all  $\hat{p}$ ).*

$p$	AP	$\sigma_{diff}$	min	max	# dec.
0.48	0.1441	0.8852	-5.5978	-15.7854	3.05
0.485	-0.0244	0.5661	-2.2687	-5.1101	2.97
0.49	0.3742	0.4914	-1.6764	3.8230	2.19
0.495	0.0526	0.3199	-1.6398	2.6498	1.29
0.5	0.1076	0.1720	0.1448	-0.1603	1.61
0.505	-0.0335	0.0874	0.1825	-0.3341	1.75

decisions made.

All measures pertain to the difference between the traditional and robust approaches.

On average, the robust strategy performs better than the traditional strategy. The best performing manager successfully reduces the volatility of realized portfolio values and captures higher maximum gains, while making less swaps than a traditional manager. Specifically, when  $p = 0.49$ , a manager implementing a robust decision-making strategy makes \$0.37 more per run on average. Further, he reduces the standard deviation of realized portfolio values by approximately 0.49 standard deviations, while making 2.19 less decisions per run on average. When  $p$  is close to 0.5, we find that the robust strategy makes decisions which closely resemble the traditional approach regarding the number of decisions made and actual swapping order between stocks, while generally earning higher returns. Also, we see that the robust model increasingly reduces the volatility of portfolio values as the volatility of the processes increases.

In cases where the robust strategy is outperformed, the difference is not very sizeable. Practically, the difference in performance will be outweighed by the amount of costs saved by making less decisions. We address this in more detail in the next section. Table 6 presents the mean results over a range of  $p$  parameters.

### 3.3.2. Heterogeneous Managers

For each value of  $\hat{p}$  in the range  $[0.5, 0.7]$ , we implement the robust decision mechanism on a set of 2000 runs, each involving a pool of 10 risky assets and one riskless asset. We hold constant the set of processes in this test, and only change the initial value of  $\hat{p}$  and the respective values for  $\epsilon$ . Results from this simulation are as follows:

We find that the manager who initially believes  $\hat{p} = 0.55$  is the top performer with respect to **AP** and **min**. We also find that he reduces the volatility of realized portfolio values, while making an average of 1.67 de-

Table 6

*Heterogeneous Managers  $\hat{p} = [0.5, 0.7]$ .*

$\hat{p}$	AP	$\sigma_{diff}$	min	max	# <sub>dec</sub>
0.5	-0.1107	0.2514	-1.0938	-0.5576	1.85
0.55	0.0684	0.2738	0.3710	2.1148	1.67
0.6	0.0024	0.2826	-0.1428	3.0338	1.74
0.65	0.0177	0.2969	-0.0137	-0.8813	1.74
0.7	-0.0860	0.2903	-0.3184	-2.9143	1.97

cisions less than a traditional manager. This simulation concentrates on a manager's initial decision, and the implications of setting inaccurate initial parameters. In line with our robust framework, the numbers suggest that limiting extreme losses and lowering the volatility of returns translates into better average performance. Managers who assume  $\hat{p}$  values closer to the median of the interval do a better job than their less conservative counterparts.

Managers who initially assume  $\hat{p} = 0.5$  or  $0.7$  fail to protect against extreme losses as effectively as the traditional strategy, while making the fewest relative decisions. Perhaps the manager who assumes an initial  $\hat{p}$  value closer to 0.5 sets his thresholds too indecisively, such that he waits too long to sell a falling stock. On the other hand, a manager who uses a traditional strategy in this case realizes his initial lack of confidence in the profitability of his risky holdings, and is quick to sell at the first signs of loss. Alternatively, perhaps the manager who assumes an initial  $\hat{p}$  value closer to 0.7 is overly confident in his risky holdings' ability to recover after realizing a mistake, which explains fewer swap decisions.

A comparison of managers' abilities to set initial parameters shows significant value in "conservative optimism," focused on limiting aggregate portfolio loss. The results in Tables 4 and 5 indicate that a tradeoff between reducing the variability of portfolio values and making less decisions is optimal.

### 3.4. Transaction Costs

After a manager decides to buy or sell a stock, he must either liquidate a current position to obtain cash to buy the desired asset, or reinvest the liquidation value of selling the stock in some other asset. The liquidation value is the market value of the sold stock  $S_t$  returned to the manager, minus the fees, expenses, or tax incurred at that time period; for instance, individual investors often must pay a certain percentage of the specific stock's

Table 5

Mean performance across all  $\hat{p} = [0.49, 0.505]$ .

$p$	AP	$\sigma_{diff}$	min	max	# dec.
$mean_{all}$	0.1252	0.2677	-0.7472	1.4946	1.71

market value times the number of shares sold. Further, investors that hold illiquid assets in their portfolio face higher potential costs of liquidating their assets, especially when performance of the assets has been poor. In mutual funds, managers are usually charged a fixed transaction penalty. Thus, we assume a total transaction cost  $C$  to be a fixed amount per share of stock sold over the entire time horizon times the number of shares sold.

Monitoring the effects of transaction costs will become increasingly necessary as the number of risky assets increases, as the number of decisions increases, and as  $T \rightarrow \infty$ . The value of accurate information becomes increasingly valuable to managers as more complexities are introduced. The results presented in Table 5 indicate that the robust strategy was outperformed by the traditional approach in two instances,  $p = 0.485$  and  $0.505$ . (Recall that  $p = 0.485$  was decided to be an unrealistic price movement magnitude to use across the entire time interval, in the sense that it yielded unrealistic values of  $u$  and  $d$ .) Dividing the relative average performance by the average number of swap decisions made in each instance, we find the average transaction cost that would make the performance of either strategy equal to zero. Specifically, the average cost per decision is \$0.008 and \$0.019 respectively. These costs are below the costs that fund managers actually incur, implying that the robust model proves profitable if implemented because it benefits from a lesser number of decisions and thus lower transaction costs than the traditional approach.

## 4. Conclusions and Future Work

### 4.1. Conclusions

In this paper, we have proposed an approach to dynamic portfolio management based on the sequential update of stock price forecasts in a robust optimization setting. Our approach takes the standpoint of an active mutual fund manager who wishes to maximize profits to individual investors. The primary contribution of this paper is the development of detailed guidelines and numerical testing of the robust decision mechanism. Numerical results suggest that our approach outperforms traditional strategies within an interval of probabilities

chosen for its practical relevance in the binomial approximation to the Lognormal model of asset prices. Our method allows us to determine the value of gaining more information about a stock process before making a final decision at the end of the time horizon.

### 4.2. Future Research Directions

This research can be extended in a number of directions. We plan to apply our methods to real market data to test the validity of our data-driven approach. Another avenue of research would consider a robust buy-in decision to complement its selling counterpart. In the case an asset has been performing well recently, but has exhibited too little or too much volatility for the investor to decide to buy right away, the investor can delay his decision until he determines the trend more precisely. An additional extension to this model is to consider making the confidence level of our thresholds a function of the current level of profit. Intuitively, this would allow an investor who has made money on his overall portfolio to be more willing to hold an asset that is experiencing a downturn in price. Alternatively, an investor who has lost money will more aggressively monitor future price movements, for fear of incurring excessive loss.

Finally, expanding the asset pool to include the trading of risky asset derivatives is an interesting consideration for our model. Because margin requirements are low in option trading, a small price movement in the wrong direction or an erroneously estimated parameter may result in losses that force an investor out of the market; on the other hand, derivatives provide relatively cheap hedging instruments. This extension, while introducing a vast amount of complexity into the problem, is a necessary step towards practical implementation.

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