Robust evaluations for duals of non-negative linear programs with box-constrained uncertainties

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Abstract. Non-negative linear programs with box-constrained uncertainties for all input data and box-constrained variables are considered. The knowledge of upper bounds for dual variables is a useful information e.g. for presolving analysis aimed at the determination of redundant primal variables. The upper bounds of the duals are found by solving a set of special continuous knapsack problems, one for each row constraint.

Key Words. Large scale linear programming, redundancy, presolving, evaluation of dual variables, robust reduction.

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1 Introduction

When a linear programming (LP) problem with non-negative coefficients has uncertain coefficients in a known range, what are the bounds where the solution of the dual problem is located? Namely what are the upper bounds for the dual variables that are valid for all of the range of uncertainty in the coefficients. This is a new problem which was not solved previously, as pointed out in (Ioslovich, 2001[a]).

LP has been a useful tool in economics, operations research, automatic control etc. for many years. In particular LP problems with non-negative coefficients play an important role, and e.g. the planning problem considered in the pioneering work (Kantorovich, 1939) was of this type. Often very large scale LP problems have to be solved. Despite extended computing capabilities there is still a large difference between solving an LP problem of intermediate dimension and of large dimension. Different methods for solving large-scale LP problems are considered in e.g. Adler *et al.,* 1991; Rogers *et al.,* 1991; Karmarkar *et al.*, 1991; Gill *et al.*, 1995; Zhang, 1996). Large LP problems almost always contain a significant number of redundant constraints and variables. This means that some constraints will never be violated and some variables will definitely be on either the zero or maximal bound. Therefore it is in general worth while to devote some effort to presolving analysis and considerably reduce the size of the problem. In this way, sometimes originally untractable problems can be solved. However, the main effect is that significant computational resources may be saved.

Various presolvers are described in (Karwan *et al.*, 1983; Brearly *et al.*, 1975; Mészáros and Suhl, 2003; Sadhana, 2002; Paulraj *et al.*, 2006; Gould and Toint, 2004). Nowadays presolvers are an integral part of many widely used LP-solvers, such as CPLEX, LIPSOL, MOSEK, and others. An interesting introduction to presolving together with important results can be found in (Gould and Toint, 2004). These presolvers play a significant role in the ability of the mentioned packages to handle very large size problems. One must keep in mind that, when pushing the button in order to solve a large scale LP, usually, by default, the first step will be a presolving procedure.

It is also well known, that the input data are not exact for most large-scale problems. This problem have been considered in (Ben-Tal and Nemirovski, 2000) where possible infeasibility and the robust counterpart problem were studied. The simple and rather usual situation is that each data item is given in some range, e.g. with a relative deviation from the given nominal value. Presolving analysis of LP problems with box-constrained uncertainty in the coefficients is treated in (Ioslovich, 1999, 2001[a,b]; Ioslovich and Gutman, 2000), where a set of algorithms is presented. In these algorithms all the evaluations are robust, meaning they are valid when the parameters of the problem (matrix and objective coefficients, values of bounds, *etc.*) are known only within some given range.

When evaluating possibly redundant columns by analysis of the dual problem, it is useful to know upper bounds of the dual variables. The computation of upper bounds is a rather complicated and previously unsolved problem. Let us consider the LP problem in the form

$$\begin{aligned} \varphi &= f'x \rightarrow max \\ Ax &\leq l, \; 0 \leq x \leq x_u, \end{aligned}$$

$$
\begin{aligned}
0 &\leq l,\ 0 \leq A,\ 0 \leq f,\ 0 \leq x_u \\
A &\in \mathbf{R^{m\times n}},\ l \in \mathbf{R^{m}},\ \ x,\ x_u,\ f \in \mathbf{R}^n
\end{aligned} \tag{1}
$$

All coefficients are assumed to be non-negative. We shall denote the rows of matrix A as a_i' and the columns as s_j. The dual problem has the form

$$
\begin{aligned}
\phi &= l'y + x_u'u \rightarrow min \\
f &\leq A'y + u, \\
0 &\leq y, \quad 0 \leq u, \\
y &\in \mathbf{R}^m,\ u \ \in \ \mathbf{R}^n.
\end{aligned} \tag{2}
$$

Here y is the vector of dual variables, related to the row constraints, and u is the vector of dual variables related to the upper bounds of the primal variables. If some of these bounds are not known, large numbers could be assigned instead. The presolving method in (Ioslovich and Makarenkov, 1975), and (Ioslovich, 1999, 2001a,b) is based on a set of tests of a single row or column. The number of calculations in the single test for one row constraint is of the same order as the problem to find a median of a set of n real numbers, (Cormen *et al.*, 1990), namely $O(n)$.

Although the non-negativity condition restricts the use of this method, the class of corresponding LP problems is still large. Let us consider the set of LP problems in the primal form (1) and in the dual form (2) with bounds of uncertainty for the data

$$
\begin{array}{lllllll}
0 & \leq & \underline{A} & \leq & A & \leq & \overline{A}, \\
0 & < & \underline{l} & \leq & l & \leq & \overline{l}, \\
0 & < & \underline{f} & \leq & f & \leq & \overline{f}, \\
0 & \leq & \underline{x}_u & \leq & x_u & \leq & \overline{x}_u,
\end{array} \tag{3}
$$

where the matrix $\underline{A}$ consists of elements $\underline{a}_{ij}$, and the matrix $\overline{A}$ consists of elements $\overline{a}_{ij}$, respectively, and where the inequalities should be interpreted componentwise. We shall denote the ith row of the matrix $\overline{A}$ as $\overline{a}_i'$, and the jth column of the same matrix as $\overline{s}_j$. We shall use notations $\underline{a}_i'$ and $\underline{s}_j$ for the ith row and jth column of the matrix $\underline{A}$, respectively. For this set of LP problems we have to find guaranteed evaluations that will allow us to detect the redundant variables and row constraints, for the given range of the input data uncertainty.

Among many examples of such LP problems we shall describe two. The first example is described in (Ioslovich, 2001[a]) and it is related to optimal production planning at a huge industrial plant. The primal variables x_j correspond to the vector of the planned amount of items to be produced, subject to upper limits x_{uj} and the "row constraints" are related to equipment, supplied raw materials, and personnel of different professions. The list of equipment and production is very huge and hence the dimensionality is very large. Moreover, the plant consists of several subdivisions, each of them generating its own constraints. The objective is to maximize the planned profit. The proposed presolving method makes it possible to reduce the problem stated in (Ioslovich, 2001[a]) from the size (15000×5000) to about (100×200).

The second example is connected with the problem of ecological monitoring and control of water quality. From its source, the water it is pumped into intermediate storage. Pumping occurs at discrete moments from different locations. For each moment and each location there exists a forecast for the concentration of a set of pollutants. The amount of water that can be pumped at each moment from each location is limited. The amount of each pollutant in the intermediate storage is strictly limited. One has to maximize the total volume of water pumped into the storage within given constraints. All the given input information has box-constrained uncertainty. The dimensionality depends on the number of monitored pollutants, time interval, number of the locations, and can be very huge. However not all the pollutants are critical at every period, therefore presolving can significantly reduce the size of this problem.

The paper is organized as follows. Sections 2 and 3 briefly summarize the main features of the presolving method described in (Ioslovich, 2001^a) in relation to redundancy of the primal variables, while Section 4 contain completely new results concerning evaluation of the upper bounds for the dual variables in the presence of box-constrained uncertainty. Numerical examples are given in Section 5.

2 General background and a principal scheme

The aim of the presolving method (Ioslovich, 2001^a) is to extract those constraints that will always be satisfied because of other constraints, and those variables that can be set in advance to its boundary values as a result of column redundancy. The scheme of the method is as follows:

A number of auxiliary small tests are performed, each of them consists of a solution of an LP problem with one row constraint and box-constrained variables, known as the *Continous knapsack problem*, (Dantzig, 1963). These tests, numerically very cheap, make it possible to evaluate the row constraints and to find and remove some of the redundant ones. In the second stage, a similar procedure is applied to the dual problem. This leads to the reduction of the number of variables (columns). Then the first stage is repeated, and the testing procedure becomes iterative. One can also note that as a result of the current reduction, the problem is decomposed into a set of smaller problems.

Finally any standard LP method can solve the problem without difficulty, because its size becomes acceptable. Computer time is significantly reduced.

Let us consider the auxiliary problem with fixed coefficients and one linear constraint

$$\begin{aligned} \psi &= f'x \rightarrow max \\ d'x &\leq b, \\ 0 &\leq x \leq x_u, \\ f &\geq 0,\ d \geq 0,\ b \geq 0. \end{aligned} \tag{4}$$

The solution of the auxiliary problem (4), which is called the "continuous knapsack problem" (CKP), was described in detail in Dantzig (1963), p. 517 (see also Appendix A). From the optimality conditions, and by denoting the dual variable for the single row constraint as ξ, it follows that

$$\begin{aligned} \forall(j : f_j < d_j\xi), x_j &= 0; \\ \forall(j : f_j > d_j\xi),\ x_j &= x_{uj}. \end{aligned} \tag{5}$$

Referring to Appendix A, the optimal solution will include the variables $x_{j_1}, x_{j_2}, \ldots, x_{j_p}$, ordered by the decreasing sequence f_j/d_j, such that

$$\sum_{k=1}^{p} d_{j_k} x_{j_k} = b \tag{6}$$

All the variables, x_{j_k} except the last one will be set to the upper limit x_{uj}. The last variable x_{j_p} which corresponds to f_{j_p}/d_{j_p}, becomes the basic variable and is included into the solution with an intermediate value,

$$0 \leq x_{j_p} \leq x_{uj_p} \tag{7}$$

The value f_{j_p}/d_{j_p} will be equal to the dual variable, ξ. If the basic variable is not equal to an intermediate value (degenerate case), then it will be assumed that the dual variable ξ is equal to f_{j_p}/d_{j_p}, where p is the last value of the sorted index that corresponds to the variable included in the solution which was set to its upper bound. Let us consider the problem of type (4) replacing the constraint $d'x \leq b$ with a single constraint of the primal problem (1), namely by the row i. This problem will have the following form

$$\begin{array}{rcl} \varphi_i & = & f'x \rightarrow max \\ a_i'x & \leq & l_i \\ 0 & \leq & x \leq x_u. \end{array} \tag{8}$$

The dual variable of this problem which is calculated similarly to ξ will be denoted as y_{ui}, and the vector of m components y_{ui} as $y_u : \; y_u \in \mathbf{R}^m$. The following theorem was proved in (Ioslovich and Makarenkov, 1975):

Theorem 1. For the pair of problems (1) - (2) and the set of problems (8) the following inequalities hold

$$y_i{}^* \leq y_{ui}, \; \forall (i = 1, ..., n), \tag{9}$$

where $y_i{}^*$ is i-th component of the optimal solution of the dual problem (2).

Using the upper bounds of the dual variables in (9) one can add box constraints to the dual problem (2). It is however not clear how to find similar upper bounds for all set of LP problems with box-constrained uncertainty in the input coefficients (3). This problem will be treated in Section 4, where *Theorem 1* will be used.

The problem (8) is aggregated, meaning that all row constraints of the problem (1) are summed with non-negative coefficients. All the coefficients of aggregation are zero except the coefficient for the constraint i which is equal to 1. The aggregated problem has an equal or greater feasible set than the feasible set of the primal problem (1). Therefore the optimal value of the objective for the aggregated problem can be used as the upper bound for the objective of the original problem (1).

We shall assume that we have obtained the value $\overline{\varphi}_l$ which is an upper bound for the objective function in all set of problems (1)-(3) such that

$$f'x \leq \overline{\varphi}_l \tag{10}$$

The algorithm how to find $\overline{\varphi}_l$ can be found in (Ioslovich, 2001^a).

3 Robust presolving analysis of dual LP problems

For the optimal values of the dual variables from (2) the following inequality holds

$$l'y + x_u'u \leq \overline{\varphi}_l. \tag{11}$$

The inequality (11) is the corollary of the equality of the optimal values of the criterion for the primal and dual problems (see Duality Theorem, Dantzig, 1963). Multiplying the inequality in (2) with the vector x_u and summing, one obtains

$$y'l_u + u'x_u \geq f'x_u. \tag{12}$$

From (11) and (12) it follows that

$$y'(l_u - l) \geq f'x_u - \overline{\varphi}_l. \tag{13}$$

From (11) it also follows that

$$l'y \leq \overline{\varphi}_l \tag{14}$$

Thus one has obtained two inequalities for the dual variables y, without the dual variables u.

For each dual problem from the set of interval inequalities (2), (3) the inequalities (11) and (12) have to be satisfied. let us denote

$$\bar{l}_u = \overline{A}\overline{x}_u$$

It follows, according to (3), that

$$\begin{array}{rcl} \underline{l}'y + x_u'u & \leq & \overline{\varphi}_l \\ y'\bar{l}_u + u'x_u & \geq & \underline{f}'\underline{x}_u. \end{array} \tag{15}$$

Summing these inequalities from (15) one obtains

$$y'(\bar{l}_u - \underline{l}) \ \geq \ \underline{f}'\underline{x}_u - \overline{\varphi}_l. \tag{16}$$

From the first inequality in (15) it also follows that

$$\underline{l}'y \ \leq \ \overline{\varphi}_l. \tag{17}$$

One can see that the inequalities (16) and (17) follow from the inequalities (2) and (3).

Using the inequality (16), and assuming that the robust upper evaluation $\overline{y}$ is known, we can solve the problem

$$\begin{array}{rcl} \underline{\eta}_{jl} & = & \underline{s}_j'y \ \rightarrow \min \\ y'(\bar{l}_u - \underline{l}) & \geq & \underline{f}'\underline{x}_u - \overline{\varphi}_l \\ 0 & \leq & y \leq \overline{y} \end{array} \tag{18}$$

Here $\overline{y}$ is the robust evaluation of upper bounds of the dual variables that will be found later in Section 4. Now we obtain the following robust test

$$\underline{\eta}_{jl} \ > \ \overline{f}_j. \tag{19}$$

If the inequality (19) is satisfied then the variable x_j must be set to zero for all the set of problems (1), (3), and the column j can be removed.
The second robust test can be obtained by solving the problem

$$\begin{array}{rcl} \overline{\eta}_{uj} & = & \overline{s}'_j y \;\rightarrow\; \max \\ \underline{l}' y & \leq & \overline{\varphi}_l \\ 0 & \leq & y \leq \overline{y}. \end{array} \tag{20}$$

This test has the form

$$\overline{\eta}_{uj} \;<\; \underline{f}_j. \tag{21}$$

If the inequality (21) is satisfied, then the variable x_j must be set to its upper bound for all the problems in the set (1), (3).

It means that the value of x_j must be at least as large as $\underline{x}_{uj}$. It is obvious that the robust evaluations (bounds) $\overline{y}$ make the feasible set of the corresponding CKP smaller, and thus improve the resulting values $\overline{\eta}$ and $\underline{\eta}$ which are used in the dual tests.

4 Robust evaluation of the dual variables

Let us denote as $\overline{y}_u$ the vector of upper bounds for y_u for all set of dual problems (2) with coefficients from (3). Recall that each component y_{ui} of y_u has been found by the solution of the correspondent CKP (8) with a single row constraint i. We have the following *Lemma 1.*

Lemma 1: For the dual variable y_i of any LP problem (1) from the set (3) the following inequality is satisfied

$$0 \leq y_i \leq \overline{y}_{ui}. \tag{22}$$

Proof: Theorem 1 holds for all LP problems (1) from the set (3). Hence the following sequence of inequalities holds

$$0 \leq y_i \leq y_{ui} \leq \overline{y}_{ui} \tag{23}$$

Thus the problem of evaluation of duals is reduced to the problem of finding the upper bound of y_{ui} for the CKP with a single row i from the set (3). Once the vector $\overline{y}_u$ is found the unknown upper bound $\overline{y}$ of all duals from (1)-(3) can be set to $\overline{y}_u$.

Now we shall show how to find $\overline{y}_{ui}$. Let us denote

$$c_{ij} = \overline{f}_j \frac{\overline{a}_{ij}}{\underline{a}_{ij}}, \tag{24}$$

and let the row vector c_i be the ith row of the matrix C. The following CKP can be considered:

$$\begin{array}{rcl} c_i x & \rightarrow & \max \\ \overline{a}_i x & \leq & \underline{l}_i, \\ 0 & \leq & x \leq \overline{x}_u. \end{array} \tag{25}$$

Solving this problem will yield the optimal value of the dual variable corresponding to the single row constraint. Let us denote this value as ζ_i. The following theorem holds:

Theorem 2: A valid choice for $\overline{y}_{ui}$ is $\overline{y}_{ui} = \zeta_i$, thus the value ζ_i is the upper bound for all y_i from (3).

Recalling the algorithm (section 2) of solving CKP, let us re-index by m the variables x_j and their coefficients when sorted according to decreasing order of the values

$$\overline{d}_j = \overline{f}_j / \underline{a}_{ij}. \tag{26}$$

Let us denote the sequence of indices $\{m\}$ as M. The value $s \in M$ is defined by the inequalities

$$\begin{aligned} \sum_{m=1}^{m=s} \overline{a}_{im}\overline{x}_{um} &\leq \underline{l}_i \\ \sum_{m=1}^{m=s+1} \overline{a}_{im}\overline{x}_{um} &> \underline{l}_i. \end{aligned} \tag{27}$$

From the algorithm it follows that $\zeta_i = \overline{f}_s / \underline{a}_{is}$. As a first step of the proof the following lemma must be formulated.

Lemma 2: Let us consider any LP problem with one row constraint i and upper bounds for variables x from the set (3). In addition to the set M (see above), the set $\underline{M}_m$ will be determined for each value of the index m as follows:

$$\underline{M}_m = \{1, 2, ..., m-1\} \tag{28}$$

For each value of the index $m \in M$, and for each value $p \in M$, and the CKP for any single row i of the set (3), it holds:

$$\forall (p : f_p/a_{ip} > \overline{f}_m / \underline{a}_{im}) \ \Rightarrow p \in \underline{M}_m \tag{29}$$

Proof: If the index value k is not a member of $\underline{M}_m$, then the following inequalities hold

$$\overline{f}_m / \underline{a}_{im} \geq \overline{f}_k / \underline{a}_{ik} \geq f_k / a_{ik} \tag{30}$$

This inequality is not satisfied for index value p and thus it belongs to $\underline{M}_m$.

Proof of *Theorem 2.* If the inequality $\zeta_i \geq y_{ui}$ is not satisfied for (at least) one of the CKP from the set (3), then for this row i there exists an index p such that $f_p/a_{ip} > \zeta_i$ which corresponds to the optimal solution of that CKP problem. According to *Lemma 2*, it belongs to the set $\underline{M}_s$ with s determined according to (27), and all indices which correspond to the variables on the upper bound also belong to the same set M_s. Thus the total number of non-zero variables is at most $s-1$ and the total value of the left side of the single constraint i, analogously to (27), is $\sum_{m=1}^{s-1} a_{im}x_u$ which is less then the corresponding sum for the upper bound, i.e. $\sum_{m=1}^{s} \overline{a}_{im}\overline{x}_{um} < \underline{l}_i$. It follows that such a dual variable cannot satisfy the optimal solution of the corresponding CKP.

5 Numerical examples

The following LP problem of form (1) is considered as a simple example:

$$\left\{ \begin{array}{lcl} f' & = & (3,\ 2,\ 1); \\ x'_u & = & (1,\ 1,\ 2); \\ A & = & \begin{pmatrix} 1 & 5 & 20 \\ 2.5 & 3 & 1 \\ 3 & 1 & 5 \end{pmatrix}; \\ l' & = & (5,\ 6.2,\ 3.8). \end{array} \right. \tag{31}$$

The solution of this LP is

$$x_1 = 1;\ x_2 = 0.8;\ x_3 = 0.$$

One can see that the optimal solution of the dual problem is non-unique and has to satisfy the conditions

$$\begin{array}{rcl} y_3 & = & 2 - 5y_1,\ y_2 = 0,\ u_2 = 0, u_3 = 0, \\ 14y_1 & = & 3 + u_1, \\ 0.4 & \geq & y_1\ \geq 3/14. \end{array} \tag{32}$$

Thus the upper bound of y_1 is attained for the solution

$$y_1 = 0.4,\ y_2 = 0,\ y_3 = 0, u_1 = 2.6.$$

Solving the set of corresponding CKP we get the vector of the upper bounds of the dual variables,

$$y_u = \{0.4,\ 0.667,\ 1.0\}.$$

A box-constrained uncertainty was added in the following way

$$d_u = 1.05; d_l = 0.95; dx_u = 1.025; dx_l = 0.975;$$

$$\overline{A} = A * d_u; \underline{A} = A * d_l;$$

$$\overline{f} = f * d_u; \underline{f} = f * d_l; \overline{l} = l * d_u;$$

$$\underline{l} = l * d_l; \overline{x_u} = x_u * dx_u; \underline{x_u} = x_u * dx_l.$$

Calculations according to *Theorem 2* give the vector of upper bounds of the dual variables, $\overline{y} = \{0.4421;\ 0.7368;\ 1.1053\}$. Within the box-constrained uncertainty, two extreme cases can be examined that give upper and lower bounds for the objective. The first is the LP $\{\underline{A},\ \overline{x}_u,\ \overline{f},\ \overline{l}\}$. This LP gives $y_u = \{0.4421,\ 0.7368,\ 0.2211\}$. The second case is the LP $\{\overline{A},\ \underline{x}_u,\ \underline{f},\ \underline{l}\}$, giving $y_u = \{0.3619,\ 0.6032,\ 0.9048\}$. One can see that in all cases the inequality $y_u \leq \overline{y}$ holds. A table of numerical results obtained for large-scale LP problems (without uncertainty in coefficients) is presented in (Ioslovich, 2001b).

Here we intend to show the results for the randomly generated problem PRIMER2000RS (size 2000×100) which contains a special modification to ensure redundancy (Ioslovich, 2001^b). The MATLAB code of the problem PRIMER2000RS is presented in Appendix B. Let $k1$ be the number of redundant rows, $k2$ the number of variables redundantly

Table 1: Results of numerical experiments

Run No	k11	k12	k13	k21	k22	k23
1	1538	12	15	1999	50	16
2	1298	10	6	1783	22	7
3	1999	47	8	1999	57	8
4	1805	26	10	1999	54	10
5	276	0	39	688	0	40
6	1999	46	15	1999	57	15
7	366	1	10	1999	48	10
8	864	1	34	1999	45	34
9	1818	30	10	1999	54	10
10	765	4	16	1999	47	17

belonging to the zero bound, and $k3$ the number of variables redundantly belonging to the upper bound. An uncertainty of 2 per cent was introduced to all input coefficients. Two iterations of the presolving algorithms were performed. The runs were performed in a single call in the MATLAB environment. In each run, MATLAB randomly generated a new LP matrix of equal size. Values of $k1, k2, k3$ after the first iteration are denoted as $k11, k12, k13$, respectively, and the corresponding values after the second iteration are denoted as $k21, k22, k23$, respectively. The results of the presolving for ten randomly generated problems are presented in Table I. The size of the problems is significantly reduced in all cases. Table I shows how many redundant rows and columns are determined in each iteration for each problem. The number of the redundant variables on the lower bound and on the upper bound are shown separately.

6 Conclusions

In this paper we have presented new results that improve presolving tests for primal and dual large scale LP problems. The evaluation of the dual variables of LP problems with uncertainty in all input data is an important element of the proposed presolving tests. An algorithm for such an evaluation is presented. The toolbox IVITEST, (Ioslovich, 2001^b), running under MATLAB contains a realization of these algorithms. The toolbox can be obtained by e-mail on request.

Appendix A

Consider the LP problem (4) noting that it has one linear row constraint only. Introduce the auxiliary vector Y with components $Y_j = d_j x_j$. Hence the constraints are

$$\sum_j Y_j \leq b,$$

and

$$0 \leq Y_j \leq Y_{uj}$$

and the objective is

$$\sum_j (f_j/d_j)Y_j \rightarrow \max$$

Now order Y by the decreasing sequence f_j/d_j,
$Y_{j1}, Y_{j2}, ..., Y_{jp}, ...$ Next chose p such that

$$\sum_{k=1}^{p-1} Y_{ujk} < b$$

and

$$\sum_{k=1}^{p} Y_{ujk} \geq b.$$

Obviously the optimal solution of (4) will be

$$\sum_{k=1}^{p} Y_{jk} = b,$$

whereby

$$Y_{jk} = Y_{ujk}, \ k = 1, ..., p-1$$

and $Y_{jp} \leq Y_{ujp}$.

Appendix B

Here is the MATLAB code of the problem PRIMER2000RS.

```
global Al Au;
Al = rand(2000,100);
[mn] = size(Al);
s = rand(10,1); a = 100*rand(1);
b = rand(1)*1e-2; c = rand(1)*1e-2;
fl = rand(100,1); fl(1:40,1) = fl(1:40,1)*c;
Al(:,41:80) = Al(:,41:80)*b;
ll = ones(size(Al(:,1))); ll(1) = ll(1)/a;
xul = ones(size(fl)); dk = 1.02;
Au = Al*dk; xuu = xul*dk;
lu = ll*dk; fu = fl*dk;
k1 = []; k2 = []; k3 = [];
for i = 1:2,
[k1,k2,k3,pm,flag,yu] =
ivitest5r(fl,fu,ll,lu,xul,xuu,k1,k2,k3);
la = size(k1); l1(i) = la(1); la = size(k2);
l2(i) = la(1); la = size(k3); l3(i) = la(1);
end;
```

The matrices $l1, l2, l3$ contain the values of the variables $k1, k2, k3$, respectively, from the first and second iterations.

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