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Hardness results and approximation algorithms for identifying codes and locating-dominating codes in graphs

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Abstract

In a graph $G = (V, E)$, an identifying code of G (resp. a locating-dominating code of G) is a subset of vertices $C \subseteq V$ such that $N[v] \cap C \neq \emptyset$ for all $v \in V$, and $N[u] \cap C \neq N[v] \cap C$ for all $u \neq v$, $u, v \in V$ (resp. $u, v \in V \smallsetminus C$), where $N[u]$ denotes the closed neighbourhood of v, that is $N[u] = N(u) \cup \{u\}$. These codes model fault-detection problems in multiprocessor systems and are also used for designing location-detection schemes in wireless sensor networks. We give here simple reductions which improve results of the paper [I. Charon, O. Hudry, A. Lobstein, Minimizing the Size of an Identifying or Locating-Dominating Code in a Graph is NP-hard, *Theoretical Computer Science 290(3) (2003), 2109–2120], and we show that minimizing the size of an identifying code or a locating-dominating code in a graph is APX-hard, even when restricted to graphs of bounded degree. Additionally, we give approximation algorithms for both problems with approximation ratio $O(\ln |V|)$ for general graphs and $O(1)$ in the case where the degree of the graph is bounded by a constant.*

Key words: approximation algorithms, approximation hardness, identifying codes, locating-dominating codes, fault tolerance, domination problems, combinatorial optimization, graph algorithms.

1. Introduction

Let $G = (V, E)$ be a simple, non-oriented graph, and for all $v \in V$ let $N(v)$ denote the neighbourhood of v, and let $N[v]$ denote the *closed neighbourhood* of v, that is : $N[v] = N(v) \cup \{v\}$. A subset of vertices $D \subseteq V$ is called a *dominating set* of G if and only if we have $N[v] \cap D \neq \emptyset$ for all $v \in V$. A subset of vertices $D_{\mathrm{t}} \subseteq V$ is called a *total dominating set* of G if and only if we have $N(v) \cap D_{\mathrm{t}} \neq \emptyset$ for all $v \in V$. A subset of vertices $C \subseteq V$ is called an *identifying code* of G if and only if it is a dominating set of G such that $N[u] \cap C \neq N[v] \cap C$ for all $u \neq v$, $u, v \in V$. A subset of vertices $D_\ell \subseteq V$ is called a *locating-dominating code* of G if and only if it is a dominating set of G such that $N[u] \cap D_\ell \neq N[v] \cap D_\ell$ for all $u \neq v$, $u, v \in V \smallsetminus D_\ell$.

If X is a locating-dominating or an identifying code of G, we usually denote $I(v, X) = N[v] \cap X$, which is called the *identifying set* of vertex v. Two vertices u and v such that $I(u, X) \neq I(v, X)$ are said to be *separated* by X, and a vertex v such that $I(v, X) \neq \emptyset$ is said to be *covered* by X.

Let us call *twins* two vertices $u \neq v$ such that $N[u] = N[v]$. A dominating set and a locating-dominating code always exist (take simply $D = D_\ell = V$), but an identifying code exists in G if and only if G has no twins. Indeed, if u and v are twins then $N[u] \cap C = N[v] \cap C$ for any subset of vertices $C \subseteq V$, and G has no identifying code; and if G has no twins then $C = V$ is a (trivial) identifying code of G. A total dominating set exists if and only if the graph has no isolated vertices, that is to say every vertex has at least one neighbour.

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The usual optimization problem associated with dominating sets (resp. total dominating sets, identifying codes, locating-dominating codes) is that of minimizing the cardinality of the respective set in a given graph. In this paper, we are interested in identifying codes and locating-dominating codes in twin-free graphs. It is known [3] that finding the minimum cardinality of an identifying code or a locating-dominating code in a graph is NP-hard.

In this paper, we derive approximation algorithms for identifying codes (Theorem 5) and locating-dominating codes (see Theorem 9). We also show that minimizing the size of a locating-dominating code is APX-hard, even when restricted to graphs of bounded degree (Theorems 6 and 7). We also derive similar results for identifying codes (see Theorems 3 and 4), and, as intermediate results, for total dominating sets (see Theorems 1 and 2). For graphs of bounded degree, we show that both problems are in APX.

Identifying and locating-dominating codes model fault-detection problems in multiprocessor systems [4,6]. Identifying codes are also used to devise indoor location-detection schemes using wireless sensor networks [7,8]. In this last application, mobile entities have to be located in an environment equipped with a network of sensors. Each entity permanently emits a signal which identifies it uniquely. The sensors are considered to deliver a binary information: a given entity is either inside or outside the range of a given sensor. Thus, each sensor dynamically knows which entities are inside its range (but no information is delivered about, say, its Euclidean distance to the sensor). The set of sensors induces then a partition of the environment into a (finite) number of subregions, according to places where ranges of sensors overlap. If the sensors are arranged so that they form an identifying code of the underlying graph, then each entity can be uniquely located in the (discretized) environment at any time. The precision of such a system is greater than the one consisting of just arranging the sensors into a dominating set.

The paper is structured as follows: the next section fixes some notations, Section 3. discusses the approximability of minimizing the size of an identifying code in a graph, Section 4. discusses the approximability of minimizing the size of a locating-dominating code in a graph, and we conclude this paper in Section 5.

2. Preliminaries

Let us define formally the optimization problems we will consider in the rest of the paper.

MIN SET COVER
Input : A family $\mathcal{F}$ of subsets of a ground set S.
Output : The minimum cardinality of a subset $C \subseteq \mathcal{F}$ such that every point of S is contained in at least one set of C.

MIN k-SET COVER
Input : A family $\mathcal{F}$ of subsets of a ground set S such that each element of $\mathcal{F}$ is of cardinality at most k.
Output : The minimum cardinality of a subset $C \subseteq \mathcal{F}$ such that every point of S is contained in at least one set of C.

MIN DOM SET
Input : A graph G.
Output : The minimum cardinality of a dominating set D of G.

MIN TOT DOM SET
Input : A graph G having no isolated vertices.
Output : The minimum cardinality of a total dominating set D_{t} of G.

MIN ID CODE
Input : A graph G having no twins.
Output : The minimum cardinality of an identifying code C of G.

MIN LOC DOM CODE
Input : A graph G.
Output : The minimum cardinality of a locating-dominating code D_ℓ of G.

We will also consider versions of these problems where the graph G will have a bounded degree $B \geq 1$, which will be denoted NAME-OF-THE-PROBLEM–B, for instance:

MIN DOM SET–B
Input : A graph G having maximum degree bounded by B.
Output : The minimum cardinality of a dominating set D of G.

In a graph G having no twins and no isolated vertices, D will denote a dominating set of G, D_ℓ will denote a locating-dominating code of G, D_{t} will denote a total dominating set of G, and C will denote an identifying code of G. We usually denote an optimal set with the superscript *, e.g. C^* will denote an identifying code of G of minimum cardinality.

We recall the notion of L-reduction (see e.g. [2]). Given two optimization problems F and G and a polynomial transformation f from instances of F to instances of G, we say that f is an *L-reduction* if there are positive constants α and β such that for every instance x of F

(1) $\mathrm{opt}_G(f(x)) \le \alpha \cdot \mathrm{opt}_F(x)$,
(2) for every feasible solution y of $f(x)$ with objective value $m_G(f(x), y) = c_2$ we can in polynomial time find a solution y' of x with $m_F(x, y') = c_1$ such that $|\mathrm{opt}_F(x) - c_1| \le \beta \cdot |\mathrm{opt}_G(f(x)) - c_2|$.

To show the APX-hardness of a problem $\mathcal{P}$, it is enough to show that there is an L-reduction from some APX-hard problem to $\mathcal{P}$ (see e.g. [2]).

3. Identifying codes

3.1. *APX-hardness of minimizing the size of an identifying code*

We use an L-reduction from MIN DOM SET–3 towards MIN TOT DOM SET–5, and then an L-reduction from MIN TOT DOM SET–5 towards MIN ID CODE–8.

Theorem 1 *The problem* MIN TOT DOM SET–B *is APX-hard for all* $B \ge 5$.

Proof : We describe an L-reduction from MIN DOM SET–3 to MIN TOT DOM SET–5. Let G be a graph on n vertices having maximum degree less than or equal to 3. Without loss of generality, we may assume that G has no isolated vertices, that is to say, each vertex has at least one neighbour. From G we construct a graph on $5n$ vertices G' by connecting the endpoints of a path $a_x b_x c_x d_x$ to each vertex x of G (see Figure 1).

Note that G' has maximum degree bounded by 5. Given a dominating set D of G, we construct a total dominating set D_{t} of G' as follows:

- D_{t} contains D,

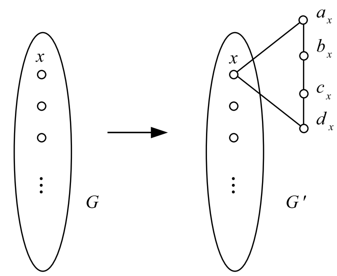

Fig. 1. Construction of G' from G. To each vertex x corresponds a path $a_x b_x c_x d_x$ whose endpoints a_x and d_x are both connected to x.

- for any vertex x of G which is not in D, the vertices b_x and c_x belong to D_{t} (see Figure 2),
- for any vertex x of D, the vertices a_x and d_x belong to D_{t} (see Figure 3),
- no other vertices belong to D_{t}.

It is straightforward to check that if D is a dominating set of G, then D_{t} is a total dominating set of G', of cardinality $|D| + 2n$. Hence

$$|D_{\mathrm{t}}^*| \ \le\ |D_{\mathrm{t}}| \ =\ |D| + 2n,$$

and since this is true for any dominating set D of G, then we have

$$|D_{\mathrm{t}}^*| \ \le\ |D^*| + 2n. \qquad (1)$$

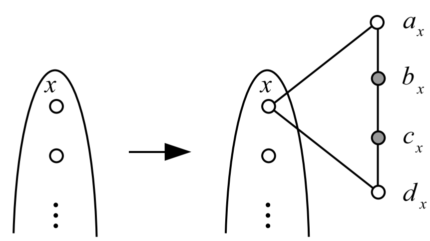

Fig. 2. For any vertex x of G which is not in D, the vertices b_x and c_x belong to D_{t}.

Conversely, let D_{t} be a total dominating set of G'. We claim that we can assume that, for each vertex x of G, exactly two vertices among a_x, b_x, c_x, d_x belong to D_{t}. Indeed, it is easy to see that at least two of these vertices belong to D_{t}, else one of them (at least) is not covered by D_{t}. Now, if at least three of them belong to D_{t}, then we can assume that only a_x, b_x and c_x belong to D_{t} (straightforward case study : if a_x, b_x, c_x and d_x belong to D_{t} then d_x can be removed, and if, say, a_x, b_x, and d_x belong to D_{t} then b_x can be removed). In this

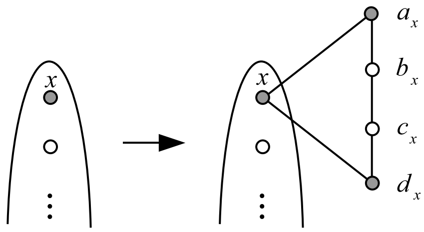

Fig. 3. For any vertex x of D, the vertices a_x and d_x belong to D_{t}.

case, we can project a_x onto a neighbour of x in G — that is to say we replace a_x by a vertex of G essentially playing the same role as a_x — and hence assume that b_x and c_x only belong to D_{t} (see Figure 4).

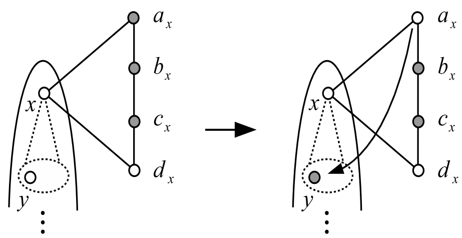

Fig. 4. If a_x, b_x, c_x belong to D_{t} (and d_x does not), then we project a_x onto a neighbour y of x in G. Indeed, if D_{t} is a total dominating set of G', then $D_{\mathrm{t}} \setminus \{a_x\} \cup \{y\}$ is a total dominating set of G' too, of cardinality less than or equal to that of D_{t}.

Now, assume that D_{t} contains exactly two vertices among a_x, b_x, c_x, d_x for each vertex x of G. It is straightforward to check that the intersection of D_{t} with G is then a dominating set of G. Indeed, for every vertex x in G which does not belong to D_{t}, we know that b_x and c_x belong to D_{t} (and a_x and d_x do not), because a_x and d_x must be covered in D_{t}. But in this case, since D_{t} is a total dominating set, then there exists in G a neighbour of x which belongs to D_{t}, and we are done. Thus, from D_{t}, we get a dominating set of G of cardinality less than or equal to $|D_{\mathrm{t}}| - 2n$, hence

$$|D^*| \ \leq \ |D_{\mathrm{t}}| - 2n.$$

Since this is true for any total dominating set D_{t}, then in particular we have

$$|D^*| \ \leq \ |D^*_{\mathrm{t}}| - 2n. \tag{2}$$

Putting (1) and (2) together, we get

$$|D^*_{\mathrm{t}}| \ = \ |D^*| + 2n.$$

Now, we are ready to prove the L-reduction. On the one hand, since G has maximum degree bounded by 3, then

$$|D| \geq \frac{n}{4}$$

for any dominating set D of G, hence

$$|D^*_{\mathrm{t}}| \ = \ |D^*| + 2n \ \leq \ 9|D^*|.$$

On the other hand, we have described a procedure which, given a total dominating set D_{t} of G', constructs a dominating set D of G such that

$$|D| \leq |D_{\mathrm{t}}| - 2n,$$

which implies

$$|D| - |D^*| \leq |D_{\mathrm{t}}| - |D^*_{\mathrm{t}}|.$$

Hence, we have an L-reduction from MIN DOM SET–3 to MIN TOT DOM SET–5 with parameters $\alpha = 9$ and $\beta = 1$. Since MIN DOM SET–3 is APX-hard [1], then MIN TOT DOM SET–5 is APX-hard, hence MIN TOT DOM SET–B is APX-hard for all $B \geq 5$. □

As a corollary, we get:

Theorem 2 *The problem* MIN TOT DOM SET *is APX-hard.*

Now, we show an L-reduction from MIN TOT DOM SET–5 towards MIN ID CODE–8.

Theorem 3 *The problem* MIN ID CODE–B *is APX-hard for all* $B \geq 8$.

Proof : We describe an L-reduction from MIN TOT DOM SET–5 to MIN ID CODE–8. Let G be a graph on n vertices having maximum degree less than or equal to 5. Without loss of generality, we may assume that G has no isolated vertices. From G we construct a graph on $4n$ vertices G' by connecting each vertex x to all the vertices of a path $a_x b_x c_x$ (see Figure 5).

Note that G' has maximum degree bounded by 8. Given a total dominating set D_{t} of G, we construct an identifying code C of G' as follows: C is composed of the union of D_{t} with all the vertices of the form a_x and c_x in G' (see Figure 6). It is straightforward to check that if D_{t} is a total dominating set of G, then C is an identifying code of G'. Hence

$$|C^*| \ \leq \ |C| \ = \ |D_{\mathrm{t}}| + 2n,$$

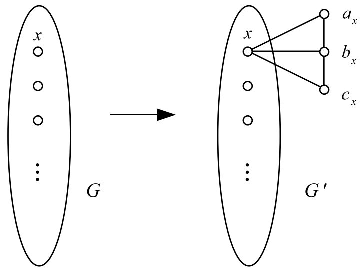

Fig. 5. Construction of G' from G. To each vertex x of G, we connect the three vertices of a path $a_x b_x c_x$.

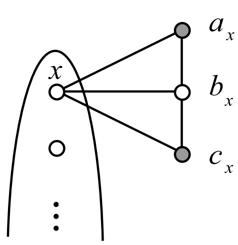

Fig. 6. C is composed of the union of D with all the vertices of the form a_x and c_x in G'.

and since this is true for any total dominating set D_{t} of G, then we have

$$|C^*| \;\le\; |D_{\mathsf{t}}^*| + 2n. \tag{3}$$

Conversely, let C be an identifying code of G'. We claim that we may assume that for each vertex x of G, a_x and c_x belong to C, and b_x does not. Indeed, since a_x must be separated from b_x, then c_x belongs to C; and, similarly, a_x must belong to C. Now, if b_x belongs to C, then we can simply remove it from C: $C \setminus \{b_x\}$ is still an identifying code of G', of smaller cardinality than C.

Now, assume that C contains a_x and c_x for each vertex x of G, and does not contain b_x. It is straightforward to check that the intersection of C with G is a total dominating set of G (because x and b_x must be separated in G'). Thus, from C, we get a total dominating set of G of cardinality less than or equal to $|C| - 2n$, hence

$$|D_{\mathsf{t}}^*| \;\le\; |C| - 2n.$$

Since this is true for any identifying code C, then in particular we have

$$|D_{\mathsf{t}}^*| \;\le\; |C^*| - 2n. \tag{4}$$

Putting (3) and (4) together, we get

$$|C^*| \;=\; |D_{\mathsf{t}}^*| + 2n.$$

Now, we are ready to prove the L-reduction. On the one hand, since G has maximum degree bounded by 5, then

$$|D_{\mathsf{t}}| \ge \frac{n}{5}$$

for any total dominating set D_{t} of G, hence

$$|C^*| \;=\; |D_{\mathsf{t}}^*| + 2n \;\le\; 11|D_{\mathsf{t}}^*|.$$

On the other hand, we have described a procedure which, given an identifying code C of G', constructs a total dominating set D_{t} of G such that

$$|D_{\mathsf{t}}| \le |C| - 2n,$$

which implies

$$|D_{\mathsf{t}}| - |D_{\mathsf{t}}^*| \le |C| - |C^*|.$$

Hence, we have an L-reduction from MIN TOT DOM SET–5 to MIN ID CODE–8 with parameters $\alpha = 11$ and $\beta = 1$. Since MIN TOT DOM SET–5 is APX-hard (from Theorem 1), then MIN ID CODE–8 is APX-hard, hence MIN ID CODE–B is APX-hard for all $B \ge 8$. □

As a corollary, we get:

Theorem 4 *The problem* MIN ID CODE *is APX-hard.*

3.2. *Positive approximation results*

Theorem 5 MIN ID CODE *is* $(2\ln|V|+1)$*-approximable, and* MIN ID CODE–B *is* $(3\ln B + 1)$*-approximable.*

Proof : Let $G = (V, E)$ be a graph, and let the *distance* between two vertices u and v, denoted by $d(u, v)$, be the minimum number of edges of a path between u and v (if such a path does not exist set $d(u, v) = \infty$, and for all $v \in V$ set $d(v, v) = 0$). Let S be the disjoint union of S_1 and S_2, where S_1 is the set of vertices of G, and S_2 is the set of all pairs of vertices of G at distance 1 or 2 from each other. Let us construct a family $\mathcal{F}$ of subsets of S as follows. Each element of $\mathcal{F}$ corresponds to a vertex $z \in V$; it contains every vertex $v \in S_1$ such that $z \in N[v]$, and it contains all pairs $(u, v) \in S_2$ such that $z \in N[u]\Delta N[v]$ (where $A\Delta B$ denotes the symmetric difference of A and B). It follows from the definitions that $C \subseteq V$ is an identifying code of G if and only if C is a solution of the MIN SET COVER problem associated with $\mathcal{F}$. Indeed, the fact that C covers all the vertices of S_1 is equivalent to the fact that C is a dominating set of G. Now, the fact that C moreover covers all the pairs of vertices in S_2 is equivalent to the fact that C is an identifying code of G. Indeed, any identifying code clearly covers all pairs of vertices in S_2. Conversely, given a dominating set C of G, two vertices u, v at distance at least 3 are necessarily such that

$$N[u] \cap C \neq N[v] \cap C$$

since their closed neighbourhoods are disjoint: $N[u] \cap N[v] = \emptyset$ for all u, v such that $d(u, v) \geq 3$. Hence, a dominating set C of G is an identifying code of G if and only if $N[u] \cap C \neq N[v] \cap C$ for all pairs of vertices u, v at distance 1 or 2 from each other.

Since MIN SET COVER is $(\ln |S| + 1)$-approximable [5], then MIN ID CODE is $(2 \ln |V| + 1)$-approximable (using the rough bound $|S| \leq |V|^2$). Furthermore, if G has bounded degree B, then each element of $\mathcal{F}$ contains at most $B + 1$ elements of S_1 and at most $B^2(B - 1)$ elements of S_2. Indeed, each vertex z clearly covers at most $B + 1$ vertices of S_1 (note that any vertex covers itself), and z separates itself from at most $B(B - 1)$ vertices (all at distance 2 from z), it separates also at most $B(B - 1)$ pairs of vertices at distance 1 (both distinct from z), and it finally separates at most $B(B - 1)(B - 2)$ pairs of vertices at distance 2 (both distinct from z). Hence if G has bounded degree B, then $(S, \mathcal{F})$ is an instance of MIN $(B^3 - B^2 + B + 1)$–SET COVER. Since MIN k–SET COVER is $(\ln k + 1)$-approximable [5], then MIN ID CODE–B is $(3 \ln B + 1)$-approximable (using the rough bound $B^3 - B^2 + B + 1 \leq B^3$, valid for all $B \geq 2$). □

4. Locating-dominating codes

4.1. *APX-hardness of minimizing the size of a locating dominating code*

Theorem 6 *The problem* MIN LOC DOM CODE–B *is APX-hard for all* $B \geq 5$.

Proof : We describe an L-reduction from MIN DOM SET–3 to MIN LOC DOM CODE–5. Let G be a graph on n vertices having maximum degree less than or equal to 3. From G we construct a graph on $3n$ vertices G' by connecting two adjacent vertices a_x, b_x to each vertex x of G (see Figure 7).

Note that G' has maximum degree bounded by 5. Given a dominating set D of G, we construct a locating-dominating code D_ℓ of G' as follows: D_ℓ is composed

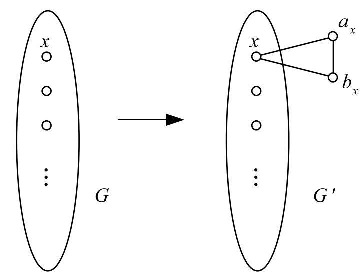

Fig. 7. Construction of G' from G. To each vertex x of G, we connect two adjacent vertices a_x and b_x.

of the union of D with all the vertices of the form a_x in G'. It is straightforward to check that if D is a dominating set of G, then D_ℓ is a locating-dominating code of G'. Hence

$$|D_\ell^*| \leq |D_\ell| = |D| + n,$$

and since this is true for any dominating set D of G, then we have

$$|D_\ell^*| \leq |D^*| + n. \tag{5}$$

Conversely, let D_ℓ be a locating-dominating set of G'. We claim that we can assume that for each vertex x of G, there is exactly one vertex a_x or b_x which belongs to D_ℓ. Indeed, if neither a_x nor b_x belongs to D_ℓ for some x, then they are not separated by D_ℓ, which contradicts the fact that D_ℓ is a locating-dominating code. Hence, at least one of them belongs to D_ℓ. Now, if both vertices a_x, b_x belong to D_ℓ, then we can either remove a_x from D_ℓ (if $D_\ell \setminus \{a_x\}$ remains a locating-dominating code of G'), or replace it by x in D_ℓ. Indeed, if $D_\ell \setminus \{a_x\}$ is no longer a locating-dominating code of G', then it means that x is not dominated in G (hence x and a_x are not separated), and in this case we can project a_x onto x and $D_\ell \setminus \{a_x\} \cup \{x\}$ is a locating-dominating code of G' (see Figure 8).

Now, assume that D_ℓ contains exactly one vertex a_x or b_x for each vertex x of G. Without loss of generality, let us assume that b_x belongs to D_ℓ for all x in G. It is straightforward to check that the intersection of D_ℓ with G is a dominating set of G (because x and a_x must be separated). Thus, from D_ℓ, we get a dominating set of G of cardinality less than or equal to $|D_\ell| - n$, hence

$$|D^*| \leq |D_\ell| - n.$$

Since this is true for any locating-dominating set D_ℓ, then in particular we have

$$|D^*| \leq |D_\ell^*| - n. \tag{6}$$

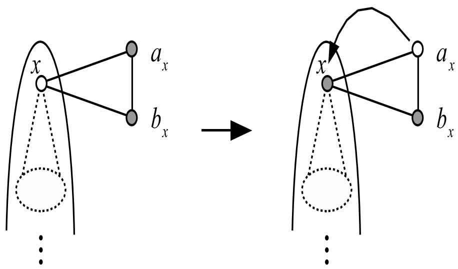

Fig. 8. For every x in G, we can assume that there is only one vertex a_x or b_x in any locating-dominating code of G'. Indeed, both vertices a_x and b_x are necessary if and only if the corresponding x is not dominated in G, and in this case we can project a_x onto x to get a locating-dominating code of G' of the same cardinality.

Putting (5) and (6) together, we get

$$|D_\ell^*| \ = \ |D^*| + n.$$

Now, we are ready to prove the L-reduction. On the one hand, since G has maximum degree bounded by 3, then

$$|D| \geq \frac{n}{4}$$

for any dominating set D of G, hence

$$|D_\ell^*| \ = \ |D^*| + n \ \leq \ 5|D^*|.$$

On the other hand, we have described a procedure which, given a locating-dominating code D_ℓ of G', constructs a dominating set D of G such that

$$|D| \leq |D_\ell| - n,$$

which implies

$$|D| - |D^*| \leq |D_\ell| - |D_\ell^*|.$$

Hence, we have an L-reduction from MIN DOM SET–3 to MIN LOC DOM CODE–5 with parameters $\alpha = 5$ and $\beta = 1$. Since MIN DOM SET–3 is APX-hard [1], then MIN LOC DOM CODE–5 is APX-hard, hence MIN LOC DOM CODE–B is APX-hard for all $B \geq 5$. □

As a corollary, we have :

Theorem 7 *The problem* MIN LOC DOM CODE *is APX-hard.*

4.2. *Positive approximation results*

We start by a result giving a relation between the sizes of locating-dominating codes and identifying codes in a graph.

Theorem 8 *Let G be a graph having no twins, let D_ℓ^* be a locating-dominating code of G of minimum cardinality, and let C^* be an identifying code of G of minimum cardinality. Then we have*

$$|D_\ell^*| \geq \frac{1}{2}|C^*|.$$

Proof : Let D_ℓ be a locating-dominating code of G. We show that there exists an identifying code C of G such that $D_\ell \subseteq C$ and $|C| \leq 2|D_\ell|$. If D_ℓ is already an identifying code of G, then we are done. If not, it means that some vertices of G are not separated by D_ℓ. Define α an equivalence relation on $V(G)$ such that $u \ \alpha \ v$ if and only if u and v are not separated by D_ℓ. Clearly, α is transitive, and $u \ \alpha \ v$ implies u and v adjacent in G. Hence, every equivalence class of α induces a complete subgraph of G. Let K be an equivalence class of α of cardinality k. We prove by induction on k that one can add at most $k-1$ vertices to D_ℓ to separate each pair of vertices of K. If $k = 1$, then we are done. Now, let us assume that $k \geq 2$, and let u and v be two vertices of K. Since G has no twins, then we may assume that there exists a vertex $z \in N[u] \smallsetminus N[v]$. This vertex z separates all pairs u', v' such that $z \in N[u']$ and $z \notin N[v']$. Therefore, adding z to K splits K into two smaller (non-empty) complete graphs, and we conclude by induction. To conclude the proof, it is enough to observe that any equivalence class of α contains at most one element of $V(G) \smallsetminus D_\ell$. □

Given an integer $n \geq 1$, let G_n be the complete graph on $2n+1$ vertices minus a maximum matching. One can show that a minimum identifying code of G_n has cardinality $2n$, whereas a minimum locating-dominating code of G_n has cardinality n. Indeed, both endpoints of any edge of the subtracted matching must belong to any identifying code, for if not one of the endpoints would not be separated from the vertex of degree $2n$ of G_n. Similarly, at least one endpoint of any edge of the subtracted matching must belong to any locating-dominating code, for if not the two endpoints would not be separated from each other. It is easy to find an identifying code (resp. a locating-dominating code) of G_n of cardinality $2n$ (resp. n). Hence, the bound of Theorem 8 is tight.

Since an identifying code of G is always a locating-dominating code of G, then we have

$$\frac{1}{2}|C^*| \ \leq \ |D_\ell^*| \ \leq \ |C^*|, \tag{7}$$

hence we deduce approximability results for locating-dominating codes :

Theorem 9 *The problem* MIN LOC DOM CODE *is* $2(2\ln|V|+1)$*-approximable, and the problem* MIN LOC DOM CODE–B *is* $2(3\ln B+1)$*-approximable.*

Proof : Straightforward from (7) and Theorem 5. □

5. Conclusion

In this paper, we presented some simple reductions improving known hardness results about minimizing the size of identifying and locating-dominating codes in graphs [3]. We also derived approximation algorithms for both problems. For graphs of bounded degree, we showed that both problems are in APX. It could be of interest to try to close the gap between the positive and the negative approximability results (between Theorems 3 and 4 and Theorem 5, between Theorems 6 and 7 and Theorem 9). To get stronger non-approximability results, one should probably reduce from another problem than MIN DOM SET, because the gap between the minimum cardinalities of a dominating set and an identifying code of a graph can be arbitrarily large (consider for example the star $K_{1,n}$, $n \geq 3$). As the problems of finding minimum identifying and locating-dominating codes in graphs remain NP-hard even when restricted to bipartite graphs [3], then it is also a natural question to ask whether one can get APX-hardness results for bipartite graphs as well.

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