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Some Necessary Conditions and a General Sufficiency Condition for the Validity of A Gilmore-Gomory Type Patching Scheme for the Traveling Salesman Problem

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Abstract

One of the most celebrated polynomially solvable cases of the TSP is the Gilmore-Gomory TSP. The patching scheme for the problem developed by Gilmore and Gomory has several interesting features. Its generalization, called the GG-scheme, has been studied by several researchers and polynomially testable sufficiency conditions for its validity have been given, leading to polynomial schemes for large subclasses of the TSP. A good characterization of the subclass of the TSP for which the GG-scheme produces an optimal solution, is an outstanding open problem of both theoretical and practical significance. We give some necessary conditions and a new, polynomially testable sufficiency condition for the validity of the GG-scheme that properly includes all previously known such conditions.

Key words: Traveling salesman problem, Gilmore-Gomory TSP, Patching Scheme, Polynomially solvable cases

1. Introduction

Given an $n \times n$ cost matrix C, the traveling salesman problem (TSP) requires finding a tour (cyclic permutation) Γ on $N = \{1, 2, \ldots, n\}$ such that its cost $c(\Gamma) = \sum_{i=1}^{n} c_{i,\Gamma(i)}$ is minimum. (Though diagonal elements of the cost matrix C do not play any role in the definition of the TSP, interestingly, many of the algorithms for polynomially solvable cases of the TSP require the diagonal elements to be finite and to satisfy specific properties. The subclass of the TSP considered in this paper is of this type.) If the cost matrix C is symmetric, then the instance of the TSP is called a *symmetric TSP* (STSP). To distinguish from this special case, the general case of the TSP is often referred to as an *asymmetric TSP* (ATSP). Throughout this paper, we deal with the general case, which we shall, for the most part, refer to as the TSP.

The TSP is a well known NP-hard problem [6] and significant literature exists on polynomially solvable special cases of it [8,14]. One of the most celebrated polynomially solvable cases of the TSP is the Gilmore-Gomory TSP [7], which can be stated as follows:

A set of n given jobs are to be heat-treated in a furnace and only one job can be treated in the furnace at any given time. The treatment of the i^{th} job involves introducing it into the furnace at a given temperature a_i and heating/cooling it in the furnace to a given temperature b_i. The costs of heating and cooling the furnace are given by functions $f(.)$ and $g(.)$, respectively. Thus, for any u, v in $\mathbb{R}$, $u < v$, the cost of heating the furnace from temperature u to temperature v is $\int_u^v f(x)dx$, while the cost of cooling the furnace from v to u is $\int_u^v g(x)dx$. Gilmore and Gomory impose the realistic condition that

$$\text{for any } x \in \mathbb{R}, \quad f(x) + g(x) \geq 0. \tag{1}$$

For each ordered pair (i, j) of jobs, if we decide to heat-treat job j immediately after job i, then the furnace temperature has to be changed from b_i to a_j. This cost, which we call the change-over cost and denote by c_{ij}, is given by

$$c_{ij} = \begin{cases} \int_{b_i}^{a_j} f(x)dx & \text{if } b_i \leq a_j \\ \int_{a_j}^{b_i} g(x)dx & \text{if } a_j < b_i \end{cases}$$

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Starting with the furnace at temperature a_1 and processing job 1 first, we want to sequentially heat-treat all the jobs and in the end return the furnace temperature to a_1. The problem is to decide the order in which the jobs should be treated in the furnace so as to minimize the total change-over cost.

Subsequent to the Gilmore-Gomory paper [7], an alternate, simple, strongly polynomial time algorithm for this special case of the TSP, with a simple proof of its validity, is given in [3] and is further extended in [12] to a larger class of problems. However, the patching scheme for the problem developed by Gilmore and Gomory in [7] is more efficient and has several interesting features; it has been further generalized to larger subclasses of the TSP in [8,14]. The most general known results in this direction are the ones in [2,15] where a generalization of the Gilmore-Gomory patching scheme, called the *GG-scheme*, is considered while fairly general polynomially testable sufficiency conditions for its validity are given, leading to polynomial schemes for large subclasses of the TSP. (See also [14].)

The GG-scheme has two main steps: (These will be described in further detail later.) (i) Choose a suitable permutation Γ on $N = \{1, 2, \ldots, n\}$. (ii) If Γ is a tour (cyclic permutation), then stop with Γ as the optimal tour. Otherwise, if Γ has $\ell > 1$ subtours, then obtain the best possible tour using a patching scheme of the following type: starting with Γ, perform a succession of $(\ell - 1)$ patching operations, where each patching operation involves choosing an $i \in N$ such that i and $(i+1)$ lie in two different subtours, breaking the two subtours by deleting the arcs, leaving nodes i and $(i + 1)$, and linking together the two resulting directed paths. Henceforth, we shall call a patching scheme of this type *GG-patching*. It is shown in [8] that the problem of choosing an optimal GG-patching is NP-hard. However, many of the well-known heuristics for the ATSP, such as those named Patch, COP in [10], which perform well in practice, can be looked upon as approximations to the GG-scheme, in which a polynomial heuristic is used to find a good GG-patching. Study of the class of the TSP for which the GG-scheme gives an optimal solution, besides being an interesting theoretical issue, also provides greater insight into the subclass of the TSP on which these heuristics perform well. As shown in [14], testing if the GG-scheme produces an optimal solution to a given instance of the TSP is an NP-hard problem. Hence, it seems unlikely that one will be able to develop polynomially testable necessary and sufficient conditions for the validity of the GG-scheme. In this paper, we give some necessary conditions and a new, more general polynomially testable sufficiency condition for the validity of the GG-scheme. What makes this result more interesting is the fairly small gap between the necessary and the sufficiency conditions. We also provide classes of the TSP which satisfy the new sufficiency conditions, but do not satisfy any of the previously known polynomially testable sufficiency conditions.

After giving our notations, definitions and some basic results in Section 2, we describe the GG-scheme in Section 3. Current results on the validity of the GG-scheme are discussed in Section 4. The main results of this paper are given in sections 5 and 6.

Most of the results in this paper were first reported in [13].

2. Notations, Definitions and Some basic results

Throughout, we assume a familiarity with the existing results on the Gilmore-Gomory TSP and its extensions. We direct the reader to [14] for details. We present in this section the main notations, definitions and the basic results that we are seeking. Additional notations used are standard ones as in [9,14].

We associate with any permutation π on N a digraph $G_\pi = [N, E_\pi]$, where $E_\pi = \{(i, \pi(i)) : i \in N\}$. Let $G_1, G_2, \ldots, G_\ell$ be the connected components of G_π with node sets $N_1, N_2, \ldots, N_\ell$, respectively. Then each G_i defines a subtour $\mathfrak{C}_i$ on the node set N_i. We call $\mathfrak{C}_1, \mathfrak{C}_2, \ldots, \mathfrak{C}_\ell$ the subtours of π. If $\ell = 1$ then π defines a tour on N and such a permutation is called a tour. If $|N_i| > 1$ then the subtour $\mathfrak{C}_i$ is called a non-trivial subtour of π. Otherwise, we call it a trivial subtour. A permutation with a single non-trivial subtour (and with all other subtours trivial) is called a circuit. A circuit with its only non-trivial subtour of the form (i, j, i) is called a transposition and is denoted by α_{ij}. A transposition of the form $\alpha_{i,i+1} = \alpha_{i+1,i}$ is called an adjacent transposition and is denoted by β_i. We denote by ξ the identity permutation (that is, $\xi(i) = i$ for all i in N). For any two permutations π and ψ on N, we define $\pi \circ \psi$ (product of π with ψ), as $\pi \circ \psi(i) = \pi(\psi(i))$ for all $i \in N$.

Observation 1 *[7] Let π be an arbitrary permutation on N and let $\{i, j\} \subseteq N$.*
(i) If i and j both belong to the same subtour $\mathfrak{C}$ of π then in $\pi \circ \alpha_{ij}$, the subtour $\mathfrak{C}$ is decomposed into two subtours, one containing i and the other containing j,

while all other subtours of $\pi \circ \alpha_{ij}$ are precisely the same as those of π.
(ii) If i and j belong to two different subtours $\mathfrak{C}_1$ and $\mathfrak{C}_2$ of π, then in $\pi \circ \alpha_{ij}$, the two subtours $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are combined into a single subtour $\mathfrak{C}$ while all other subtours of $\pi \circ \alpha_{ij}$ are precisely the same as those of π.

In case (ii) of Observation 1, we say that subtour $\mathfrak{C}$ is obtained by patching the subtours $\mathfrak{C}_1$ and $\mathfrak{C}_2$. If, in addition, $j = i + 1$, then we call it an adjacent patching scheme. Starting with a permutation Γ on N with ℓ subtours, the GG-patchings in the GG-scheme is a sequence of $(\ell - 1)$ adjacent patchings, which result in a tour.

The following concept of pyramidal tours introduced in [1] plays an important role in the study of a GG-scheme.

Definition 1. [1] A path in a digraph $G = [N, E]$ is said to be a pyramidal path if and only if it is of the form $(i_1, i_2, \ldots, i_u, j_1, j_2, \ldots, j_v)$ with $i_1 < i_2 < \cdots < i_u$ and $j_1 > j_2 > \cdots > j_v$. A closed, pyramidal path is called a pyramidal subtour. A permutation is said to be pyramidal if and only if all its non-trivial subtours are pyramidal. An instance of the TSP, TSP(C), is said to be pyramidally solvable if and only if it has an optimal tour which is pyramidal.

It may be noted that whether a path is pyramidal depends on the numbering of the nodes. For a given node numbering, an optimal pyramidal tour can be computed in $O(n^2)$ time [17]. However, testing if, for a given node numbering, the given instance of the TSP is pyramidally solvable is an NP-hard problem [14] and various polynomially testable sufficiency conditions for it are reported in relevant literature [14].

Definition 2. [5] A permutation is said to be dense if and only if the node set of each of its non-trivial subtours is of the form $\{i, i+1, \ldots, j\}$. Let π be a dense permutation with its non-trivial subtours $\mathfrak{C}_1$, $\mathfrak{C}_2$, $\ldots$, $\mathfrak{C}_\ell$ on node sets $\{i_1, i_1+1, \ldots, j_1\}$, $\{i_2, i_2+1, \ldots, j_2\}$, $\ldots$, $\{i_\ell, i_\ell+1, \ldots, j_\ell\}$, respectively. Then we say that π is dense on node set $\{i_1, i_1+1, \ldots, j_1-1\} \cup \{i_2, i_2+1, \ldots, j_2-1\} \cup \ldots \cup \{i_\ell, i_\ell+1, \ldots, j_\ell-1\}$.

The relationship between GG-patchings and pyramidal tours is established by the following lemma.

Lemma 1 *For any set $S = \{u, u+1, \ldots, v\} \subseteq \{1, 2, \ldots, n-1\}$, $k = |S|$ and any ordering $(i_1, i_2, \ldots, i_k)$ of elements of S, permutation $\psi = \beta_{i_1} \circ \beta_{i_2} \circ \cdots \circ \beta_{i_k}$ is a pyramidal circuit dense on set S. Conversely, for any pyramidal circuit ψ, dense on S, there exists an ordering $(i_1, i_2, \ldots, i_k)$ of elements of S such that $\psi = \beta_{i_1} \circ \beta_{i_2} \circ \cdots \circ \beta_{i_k}$.*

Definition 3. For any $N \supseteq X = \{i_1, i_2, \ldots, i_k\}$, where $1 \le i_1 < i_2 < \cdots < i_k \le n$,
(i) $[i_1, i_k - 1]$ is the range of X.
(ii) For each $1 \le u < k$, $\{i_u, i_u + 1, \ldots, i_{u+1} - 1\}$ is a region of X.
(iii) If X is the node set of a subtour $\mathfrak{C}$, then we call the range and the regions of X as, respectively, the range and the regions of $\mathfrak{C}$.

Definition 4. Suppose digraph G_π, associated with a permutation π, has ℓ connected components with node sets $N_1, N_2, \ldots, N_\ell$. Then $G_p^\pi = [N_p^\pi, E_p^\pi]$, the patching pseudograph of π, is defined as $N_p^\pi = \{1, 2, \ldots, \ell\}$ and $E_p^\pi = \{e_i = (u, v) : i \in \{1, 2, \ldots, n-1\}, i \in N_u, (i+1) \in N_v\}$. For any $S \subseteq \{1, 2, \ldots, n-1\}$ we denote by $E_p^\pi[S]$ the set $\{e_i \in E_p^\pi : i \in S\}$.

It is observed in [7] that for a permutation π on N with $\ell > 1$ connected components, and a set $S = \{i_1, i_2, \ldots, i_{(\ell-1)}\} \subseteq N$, $\pi \circ \beta_{i_1} \circ \beta_{i_2} \circ \cdots \circ \beta_{i_{(\ell-1)}}$ is a tour if and only if $E_p^\pi[S]$ is the edge set of a spanning tree of G_p^π.

For an $n \times n$ cost matrix C, we denote its $(i,j)^{th}$ element by $c_{i,j}$ and for any permutation ψ on N, we define the cost of ψ as

$$c(\psi) = \sum_{i=1}^{n} c_{i,\psi(i)}$$

The traveling salesman problem is then to find a tour Γ on $N = \{1, 2, \ldots, n\}$ such that $c(\Gamma)$ is minimum.

Definition 5. For any $n \times n$ cost matrix C with finite entries (including the diagonal entries), the density matrix D of C is an $(n-1) \times (n-1)$ matrix defined as

$$d_{ij} = c_{i,j+1} + c_{i+1,j} - c_{ij} - c_{i+1,j+1} \quad \forall\ 1 \le i, j < n.$$

For example, the density matrix of
$C = \begin{bmatrix} 2 & 4 & 1 \\ 3 & 6 & 5 \\ 4 & 5 & 3 \end{bmatrix}$ is $D = \begin{bmatrix} -1 & -2 \\ 2 & 1 \end{bmatrix}$.

Definition 6. For any cost matrix C and any two permutations π and ψ on N, we define the permuted cost matrix $C^{\pi,\psi}$ as

$$c_{ij}^{\pi,\psi} = c_{\pi(i),\psi(j)} \quad \forall\ i, j.$$

We denote $C^{\xi,\psi}$ by C^ψ. (It may be recalled that ξ denotes the identity permutation.) Thus, $c(\pi \circ \psi) = c^\pi(\psi) = \sum_{i \in N} c_{i,\psi(i)}^\pi$.

We define the cost of ψ relative to π as

$$c(\pi \circ \psi) - c(\pi) = c^{\pi}(\psi) - c^{\pi}(\xi).$$

As shown in [2,15], the cost of ψ relative to π depends only on the density matrix D of C^{π} and we denote it by $D(\psi)$. Let F be the set of all orderings of the elements of S. We define,

$$D[S] = \min\{D(\beta_{i_1} \circ \beta_{i_2} \circ \cdots \circ \beta_{i_k}) : (i_1, i_2, \ldots, i_k) \in F\}. \quad (2)$$

Observation 2. *For any $S \subseteq \{1, 2, \ldots, n-1\}$, let $(S_1 \cup S_2 \cup \cdots \cup S_\ell)$ be its natural partition. (That is, for any $x \in \{1, 2, \ldots, \ell\}$, S_x is of the form $\{i_x, i_x+1, \ldots, j_x\}$ and for all $x \in \{1, 2, \ldots, \ell-1\}$, $j_x + 1 < i_{x+1}$.) Then $D[S] = \sum_{i=1}^{\ell} D[S_i]$.*

3. GG-scheme : A generalization of the Gilmore-Gomory patching scheme

The following generalization, of the Gilmore-Gomory patching scheme (called the GG-scheme) is studied in [2,4,5,11,15,16]. (See also [8,14].)

Algorithm 1 *GG-Scheme*
Input: An $n \times n$ cost matrix C and a suitable permutation Γ on $N = \{1, 2, \ldots, n\}$
Step 1: If Γ is a tour, then stop with Γ as the output. Otherwise, let ℓ be the number of subtours of Γ and let $N_1, N_2, \ldots, N_\ell$ be the node sets of these subtours.
Step 2: Construct the patching pseudograph $G_p^{\Gamma} = [N_p^{\Gamma}, E_p^{\Gamma}]$ of Γ.
Step 3: Compute the density matrix D of C^{Γ}. Find a spanning tree in G_p^{Γ} with an edge set, say $\{e_i : i \in T^*\}$, such that $D[T^*]$ is minimum.
Let $(i_1, i_2, \ldots, i_{\ell-1})$ be an ordering of the elements of T^* such that $D[T^*] = D(\beta_{i_1} \circ \beta_{i_2} \circ \cdots \circ \beta_{i_{\ell-1}})$. Let $\Psi = \beta_{i_1} \circ \beta_{i_2} \circ \cdots \circ \beta_{i_{\ell-1}}$. Construct the tour $\Gamma^* = \Gamma \circ \Psi$. Output Γ^* and stop.

4. Existing sufficiency results for the validity of the GG-scheme

As shown in [14], checking if, for a given pair (C, Γ), the GG-scheme, with Γ as the suitable permutation, produces an optimal tour is NP-hard even for the special case $\Gamma = \xi$. Hence, it seems unlikely that one will be able to find polynomially testable necessary and sufficient conditions on an $(n-1) \times (n-1)$ matrix D under which, for any pair (C, Γ) of a cost matrix C and a permutation Γ on N such that D is the density matrix of C^{Γ}, the GG-scheme with Γ as the suitable starting permutation will produce an optimal tour.

The most generally known such polynomially testable sufficiency condition on D is the one given in [2,15] and is a special case of the condition in Theorem 3 below, which is a minor modification of a theorem in [2,15].

For any $n \times n$ matrix C, with a density matrix D and any $1 \le i, j < n$ and $1 \le u, v < n$, we denote

$$M^D_{i,j,u,v} = \sum_{y=u}^{v} \sum_{x=i}^{j} d_{xy}. \quad (3)$$

From the definition of a density matrix, it follows that,

$$M^D_{i,j,u,v} = \begin{cases} c_{i,v+1} + c_{j+1,u} - c_{i,u} - c_{j+1,v+1}, & \text{if } i \le j \text{ and} u \le v \\ 0, & \text{otherwise} \end{cases}$$

Definition 7. For any permutation Ψ on N with non-trivial subtours
$\mathfrak{C}_1, \mathfrak{C}_2, \ldots, \mathfrak{C}_\ell$, having respective ranges $[i_1, j_1 - 1]$, $[i_2, j_2 - 1], \ldots, [i_\ell, j_\ell - 1]$, the intersection graph of the non-trivial subtours of Ψ is the graph
$G_{\Psi}^{I} = [N_{\Psi}, E_{\Psi}^{I}]$, where $N_{\Psi} = \{1, 2, \ldots, \ell\}$ and
$E_{\Psi}^{I} = \{(i, j) : 1 \le i, j \le \ell, i \ne j;$ ranges of the subtours $\mathfrak{C}_i$ and $\mathfrak{C}_j$ intersect $\}$.

Theorem 2 *[14] Suppose that $\Gamma \circ \Psi$ is a tour. Suppose Ψ has ℓ non-trivial subtours and G_{Ψ}^{I} has r connected components. Let $N_{\Psi}^{1}, N_{\Psi}^{2}, \ldots, N_{\Psi}^{r}$ be the node sets of the r connected components of G_{Ψ}^{I}. For each $i \in \{1, 2, \ldots, r\}$, let $|N_{\Psi}^{i}| = \ell_i$ and let X_{Ψ}^{i} be the union of the node sets of all the non-trivial subtours of Ψ corresponding to the nodes in N_{Ψ}^{i}. Then, there exists $S \subseteq \{1, 2, \ldots, n-1\}$ and a partition $\{S_1, S_2, \ldots, S_r\}$ of its elements such that:*
(i) $E_p^{\Gamma}[S] = \{e_i \in E_p^{\Gamma} : i \in S\}$ is the edge set of a spanning tree of G_p^{Γ}.
(ii) For $1 \le i \le r$, every element of S_i lies in the range of X_{Ψ}^{i}; and where every region of X_{Ψ}^{i} contains, at most, one element of S_i; and where $|S_i| \le (|X_{\Psi}^{i}| - \ell_i)$.
(iii) $|S| \equiv ((\sum_{i=1}^{r} |X_{\Psi}^{i}|) - \ell) mod 2$.

Theorem 3 *Suppose D is an $(n-1) \times (n-1)$ matrix satisfying the following condition:*

Let Ψ be any arbitrary permutation on N. Suppose Ψ has ℓ non-trivial subtours and G_{Ψ}^{I} has r connected components of sizes $\ell_1, \ell_2, \ldots, \ell_r$. Let $X_{\Psi}^{1}, X_{\Psi}^{2}, \ldots, X_{\Psi}^{r}$ be the unions of the node sets of the *non-trivial subtours of Ψ corresponding, respectively, to nodes of the r connected components of G^I_Ψ. Let S be any subset of N with a partition $\{S_1, S_2, \ldots, S_r\}$, such that (i) for any $1 \leq i \leq r$, every element of S_i lies in the range X^i_Ψ; and where every region of X^i_Ψ contains, at most, one element of S_i; and where $|S_i| \leq (|X^i_\Psi| - \ell_i)$; and (ii) $|S| \equiv ((\sum_{i=1}^r |X^i_\Psi|) - \ell)$mod2. Then $D(\Psi) \geq D[S]$.*

Under this condition, for any cost matrix C and any permutation Γ on N such that D is the density matrix of C^Γ, the GG-scheme with Γ as the suitable starting permutation produces an optimal solution to the corresponding TSP.

Proof: Let D be an $(n-1) \times (n-1)$ matrix satisfying the condition of the theorem and let a cost matrix C and a permutation Γ on N be such that D is the density matrix of C^Γ. Let Υ be an optimal tour to the corresponding instance of the TSP. Let $\Psi = \Gamma^{-1} \circ \Upsilon$. Suppose Ψ has ℓ non-trivial subtours and G^I_Ψ has r connected components of sizes $\ell_1, \ell_2, \ldots, \ell_r$. Let $X^1_\Psi, X^2_\Psi, \ldots, X^r_\Psi$ be the unions of the node sets of the non-trivial subtours of Ψ corresponding respectively to the r connected components of G^I_Ψ. Then, by Theorem 2, there exist a subset S of N with partition $\{S_1, S_2, \ldots, S_r\}$ such that:(i) for each $1 \leq i \leq r$, every element of S_i lies in the range X^i_Ψ; and where every region of X^i_Ψ contains, at most, one element of S_i; and where $|S_i| \leq (|X^i_\Psi| - \ell_i)$; (ii) $E^\Gamma_p[S]$ is the edge-set of a spanning tree of the patching pseudograph G^Γ_p; and, therefore, (iii) $|S| \equiv ((\sum_{i=1}^r |X^i_\Psi|) - \ell)$mod2. Let $(i_1, i_2, \ldots, i_r)$ be an ordering of the elements of S for which $D(\beta_{i_1} \circ \beta_{i_2} \circ \cdots \circ \beta_{i_r}) = D[S]$. Then, $\tau^* = \Gamma \circ \beta_{i_1} \circ \beta_{i_2} \circ \cdots \circ \beta_{i_r}$ is a tour on N; and $c(\tau^*) - c(\Upsilon) = (c(\tau^*) - c(\Gamma)) - (c(\Upsilon) - c(\Gamma)) = D[S] - D(\Psi) \leq 0$. Hence, τ^* is an optimal tour. This proves the Theorem.

5. Some necessary conditions for the validity of the GG-scheme

In this section, we investigate necessary conditions on an $(n-1) \times (n-1)$ matrix D, for which, for any cost matrix C, and any permutation Γ on N such that D is the density matrix of C^Γ, the GG-scheme with Γ, as the suitable permutation, produces an optimal tour.

The following examples dispel a commonly held false belief that it is a necessary condition for Γ to be an optimal solution to the corresponding Assignment problem on C.

Example 1 [14]: Let D be a 2×2 matrix with $d_{1,1} = -1$ with each of the other three entries equal to 5. Then it is easy to see that for any 3×3 matrix C and permutation Γ on $\{1, 2, 3\}$, such that D is the density matrix of C^Γ, $\Gamma \circ \beta_1$ is the unique optimal solution to the Assignment problem on C and the GG-scheme, with Γ as the suitable permutation, produces an optimal tour.

Example 2: Consider a TSP with the following cost matrix C:

$$\begin{bmatrix} 3 & 0 & 15 & 9 & 11 \\ 0 & 2 & 17 & 11 & 13 \\ 4 & 1 & 6 & 0 & 2 \\ 17 & 9 & 4 & 3 & 0 \\ 33 & 20 & 0 & 9 & 1 \end{bmatrix}$$

Consider a non-optimal assignment $\Gamma = (1,1)\,(2,2)\,(3,4,5,3)$. The permuted matrix C^Γ is shown below:

$$\begin{bmatrix} 3 & 0 & 9 & 11 & 15 \\ 0 & 2 & 11 & 13 & 17 \\ 4 & 1 & 0 & 2 & 6 \\ 17 & 9 & 3 & 0 & 4 \\ 33 & 20 & 9 & 1 & 0 \end{bmatrix}$$

The density matrix D of C^Γ is shown below:

$$\begin{bmatrix} -5 & 0 & 0 & 0 \\ 5 & 10 & 0 & 0 \\ 5 & 5 & 5 & 0 \\ 5 & 5 & 5 & 5 \end{bmatrix}$$

The optimal assignment, $\Gamma^* = (1,2,1)\,(3,4,5,3)$ is non-diagonal, unique and has two subtours $(1,2,1)$ and $(3,4,5,3)$. The cost of the optimal assignment is 0. The GG subtour patching scheme, starting with optimal assignment Γ^*, finds only one candidate tour $(1,2,4,5,3,1)$, which has a cost of 15. However, the GG-scheme starting with the non-optimal assignment Γ finds two candidate tours and picks the unique optimal tour $(1,4,5,3,2,1)$, which has a cost of 10. Thus, the example shows that the GG-scheme may fail to obtain an optimal tour when the initial assignment is an optimal assignment. But, the GG-scheme obtains an optimal tour when the initial assignment is not an optimal assignment at all.

Theorem 4 *Each of the following conditions on an $(n-1) \times (n-1)$ matrix D is a necessary condition for the GG-scheme, with a permutation Γ on N as the suitable permutation, to produce an optimal tour for an instance of the TSP with a cost matrix C, where C and Γ are such that D is the density matrix of C^Γ:*
(i) $\forall 1 \leq i, j \leq n-1$, *such that* $|i-j| > 1$, $d_{ii} + d_{jj} \geq 0$;
(ii) $\forall 1 \leq i \leq n-2$, $D[\{i, i+1\}] \geq 0$;
(iii) $\forall 1 \leq i < j \leq n-1$, $M^D_{i,j,i,j} \geq d_{uu} \forall i \leq u \leq j$;
(iv) $\forall 1 \leq i < j \leq n-1$, $M^D_{i,j,i,j} \geq 0$;

(v) $\forall 1 \leq i \leq j < j+1 \leq k \leq n-1$
$M^D_{i,j,i,j} + M^D_{j+1,k,j+1,k} + \min\left(M^D_{i,j,j+1,k}, M^D_{j+1,k,i,j}\right) \geq d_{jj} + d_{j+1,j+1} + \min\left(d_{j,j+1}, d_{j+1,j}\right) \geq 0$*;*
(vi) $\forall 1 \leq i \leq j < j+1 \leq k \leq n-1$
$M^D_{i,j,i,j} + M^D_{j+1,k,j+1,k} + \min\left(M^D_{i,j,j+1,k}, M^D_{j+1,k,i,j}\right) \geq d_{uu} + d_{vv} \geq 0 \forall i \leq u \leq j < j+1 \leq v \leq k$, $\forall u+1 < v$*;*
(vii) For any circuit φ with its non-trivial cycle on node set $\{i, u, u+1, \ldots, v, j\}$, $1 \leq i < u \leq v < j \leq n$, and any optimal pyramidal and dense permutation β with its non-trivial cycle on node set $\{u-1, u, u+1, \ldots, v, v+1\}$, $D(\varphi) \geq D(\beta) \geq 0$;
(viii) For any principal submatrix D' of D on a consecutive subset of its rows and columns, and any cost matrix C having D' as its density matrix, the TSP on C is pyramidally solvable;
(ix) At most, two diagonal entries are negative. If two diagonal entries are negative, they must be consecutive. If $d_{ii} < 0$ and $d_{i+1,i+1} < 0$ for some i, then at least one of $\Gamma \circ \beta_i$ and $\Gamma \circ \beta_{i+1}$ is an optimal assignment. If exactly one diagonal entry is negative, $\Gamma \circ \beta_i$ is an optimal assignment, where $d_{ii} < 0$ for some i. If all $d_{ii} \geq 0$, then Γ is an optimal assignment.

Proof: Consider an $(n-1)\times(n-1)$ matrix D for which for any C and Γ, such that D is the density matrix of C^Γ, the GG-scheme with Γ as the suitable permutation produces an optimal tour.

(i) Suppose $1 \leq i < j-1 \leq n-2$, such that $d_{ii} + d_{jj} < 0$. Define Γ as follows: $\Gamma(i) = j$; $\Gamma(j) = i+1$; $\Gamma(j-1) = j+1$; $\Gamma(n) = 1$; and $\Gamma(\ell) = \ell+1$ for all other $1 \leq \ell \leq n$. Let C be any cost matrix such that D is the density matrix of C^Γ. Then Γ is a tour on N and therefore the GG-scheme will terminate with Γ as the output. But $\Gamma \circ \beta_i \circ \beta_j$ is a tour having a strictly lower cost than Γ. We thus have a contradiction.

(ii) Suppose $1 \leq i \leq n-2$, such that $D[\{i, i+1\}] < 0$. Define Γ as follows: $\Gamma(n) = 1$; and $\Gamma(j) = j+1$ for all $1 \leq j < n$. Let C be any cost matrix such that D is the density matrix of C^Γ. Then Γ is a tour on N and therefore the GG-scheme will terminate with Γ as the output. But at least one of the tours $\Gamma \circ \beta_i \circ \beta_{i+1}$ and $\Gamma \circ \beta_{i+1} \circ \beta_i$ has a strictly lower cost than Γ. We thus have a contradiction.

(iii) Suppose $1 \leq i < j \leq n-1$, $i \leq u \leq j$, such that $M^D_{i,j,i,j} < d_{uu}$. Define Γ as follows: $\Gamma(u) = 1$; $\Gamma(n) = u+1$; $\Gamma(\ell) = \ell+1$ for all other $1 \leq \ell \leq n$. Then, for any cost matrix C such that D is the density matrix of C^Γ, the GG-scheme with Γ as the suitable starting permutation produces the tour $\Gamma \circ \beta_u$ as the output. But the tour $\Gamma \circ \alpha_{i,j}$ has a strictly lower cost than $\Gamma \circ \beta_u$. We thus have a contradiction.

(iv) Suppose $1 \leq i < j \leq n-1$, such that $M^D_{i,j,i,j} < 0$. Then it follows from (i) and (iii) that $j = i+1$. Now, by (ii), $d_{i,i+1} \geq -(d_{ii} + d_{i+1,i+1})$ and $d_{i+1,i} \geq -(d_{ii} + d_{i+1,i+1})$. Therefore, $M^D_{i,i+1,i,i+1} \geq -(d_{ii} + d_{i+1,i+1}) > 0$. We thus have a contradiction.

(v) Suppose $1 \leq i \leq j < j+1 \leq k \leq n-1$, such that $M^D_{i,j,i,j} + M^D_{j+1,k,j+1,k} + \min\left(M^D_{i,j,j+1,k}, M^D_{j+1,k,i,j}\right) < d_{jj} + d_{j+1,j+1} + \min\left(d_{j,j+1}, d_{j+1,j}\right)$. Define Γ as follows: $\Gamma(j) = 1; \Gamma(j+1) = j+1; \Gamma(n) = j+2$; $\Gamma(\ell) = \ell+1$ for all other $1 \leq \ell \leq n$. Then, for any cost matrix C such that D is the density matrix of C^Γ, the GG-scheme with Γ as the suitable starting permutation produces the tour $\Gamma \circ \beta_j \circ \beta_{j+1}$ or $\Gamma \circ \beta_{j+1} \circ \beta_j$ as the output. But at least one of the tours $\Gamma \circ \alpha_{i,j} \circ \alpha_{j+1,k}$ or $\Gamma \circ \alpha_{j+1,k} \circ \alpha_{i,j}$ has a strictly lower cost than both tours $\Gamma \circ \beta_j \circ \beta_{j+1}$ and $\Gamma \circ \beta_{j+1} \circ \beta_j$. We thus have a contradiction. The non-negativitiy part of condition (v) follows from (ii).

(vi) Suppose$1 \leq i \leq j < j+1 \leq k \leq n-1$, $i \leq u \leq j < j+1 \leq v \leq k, u+1 < v$ such that $M^D_{i,j,i,j} + M^D_{j+1,k,j+1,k} + \min\left(M^D_{i,j,j+1,k}, M^D_{j+1,k,i,j}\right) < d_{uu} + d_{vv}$. Define Γ as follows: $\Gamma(u) = 1; \Gamma(v) = u+1$; $\Gamma(n) = v+1$; $\Gamma(\ell) = \ell+1$ for all other $1 \leq \ell \leq n$. Then, for any cost matrix C, such that D is the density matrix of C^Γ, the GG-scheme with Γ as the suitable starting permutation produces the tour $\Gamma \circ \beta_u \circ \beta_v$ or $\Gamma \circ \beta_v \circ \beta_u$ as the output. But at least one of the tours $\Gamma \circ \alpha_{i,j} \circ \alpha_{j+1,k}$ or $\Gamma \circ \alpha_{j+1,k} \circ \alpha_{i,j}$ has a strictly lower cost than both tours $\Gamma \circ \beta_u \circ \beta_v$ and $\Gamma \circ \beta_v \circ \beta_u$. We thus have a contradiction. The non-negativitiy part of condition (vi) follows from (i).

(vii) Suppose there exists a circuit φ with its non-trivial cycle on node set $\{i, u, u+1, \ldots, v, j\}$, $1 \leq i < u \leq v < j \leq n$, and an optimal pyramidal and dense permutation β with its non-trivial cycle on node set $\{u-1, u, u+1, \ldots, v, v+1\}$, such that $D(\varphi) < D(\beta)$. Define Γ as follows: $\Gamma(u-1) = 1; \Gamma(n) = v+1; \Gamma(\ell) = \ell\ \forall u \leq \ell \leq v$; $\Gamma(\ell) = \ell+1$ for all other $1 \leq \ell \leq n$. Then, for any cost matrix C, such that D is the density matrix of C^Γ, the GG-scheme with Γ as the suitable starting permutation produces the tour $\Gamma \circ \beta$ as the output. But the tour $\Gamma \circ \varphi$ has a strictly lower cost than the tour $\Gamma \circ \beta$. We thus have a contradiction. Now suppose $D(\beta) < 0$. If $(v-u)$ is odd, there exists at least one k, such that $(u-1) \leq k \leq v$ and $d_{kk} > 0$. Define Γ as follows: $\Gamma(k) = 1$; $\Gamma(n) = (k+1)$; and $\Gamma(\ell) = \ell + 1$ for all $1 \leq \ell < n$. Let C be any cost matrix such that D is the density matrix of C^Γ. Then the GG-scheme, with Γ as the suitable starting permutation, produces the tour $\Gamma \circ \beta_k$ as the output. But the tour $\Gamma \circ \beta$ has a strictly lower cost than $\Gamma \circ \beta_k$. We thus have a contradiction. If $(v-u)$ is not odd, define Γ as follows: $\Gamma(n) = 1$; and $\Gamma(\ell) = \ell + 1$ for all $1 \leq \ell < n$. Let C be any cost matrix such that D is the density matrix of C^Γ. Then the GG-scheme, with Γ as the suitable starting permutation, will terminate with Γ as the output. But the tour $\Gamma \circ \beta$ has a strictly lower cost than Γ. We thus have a contradiction.

(viii) For any $1 \leq i < j \leq n-1$, consider the set $S = \{i, i+1, \ldots, j\}$. Let D' be the principal submatrix of D on row/column set S. Define a permutation Γ on N as follows: $\Gamma(i) = 1$; $\Gamma(n) = j+1$; $\Gamma(\ell) = \ell$ $\forall i < \ell \leq j$; $\Gamma(\ell) = \ell + 1$ for all other $\ell \in N$. Let Υ be the tour produced by the GG-scheme on D with Γ as the initial suitable permutation. Then $\Omega = \Gamma^{-1} \circ \Upsilon$ is a circuit with its only non-trivial subtour a pyramidal one on the node set $\{i, i+1, \ldots, j+1\}$. Furthermore, it is easy to see that for any circuit φ on N with its only nontrivial subtour on node set $\{i, i+1, \ldots, j+1\}$, $\Gamma \circ \varphi$ is a tour. Since the GG-scheme works on D, we must have $D(\Omega) \leq D(\varphi)$. Hence, Ω is an optimal tour for any cost matrix C' with D' as its density matrix.

(ix) If there are more than two diagonal entries negative, there is at least one pair of non-consecutive diagonal entries and (i) does not hold. Hence, there may be, at most, two consecutive negative diagonal entries or just one negative diagonal entry or none. If condition (ix) is false, there exists a permutation Ψ such that $\Gamma \circ \Psi$ is an optimal assignment and Ψ has at least one subtour $\mathfrak{C}$ such that $D(\mathfrak{C}) < 0$. It follows from (i)-(iv) that $|\mathfrak{C}| \geq 3$. For some $1 \leq i_1 < i_2 < \ldots < i_k \leq n$, let $S = \{i_1, i_2, \ldots, i_k\}$ be the node set of subtour $\mathfrak{C}$. Define Γ as follows: $\Gamma(i_1) = 1$; $\Gamma(n) = i_{k-1} + 1$; $\Gamma(i_j) = i_{j-1} + 1$ $\forall 2 \leq j \leq (k-1)$; $\Gamma(\ell) = \ell + 1$ for all other $1 \leq \ell \leq n$. Let β be an optimal pyramidal and dense permutation on node set $\{i_1, i_2, \ldots, i_{k-1}\}$. Then, for any cost matrix C, such that D is the density matrix of C^Γ, the GG-scheme with Γ as the suitable starting permutation produces the tour $\Gamma \circ \beta$ as the output. But the tour $\Gamma \circ \mathfrak{C}$ has a strictly lower cost than the tour $\Gamma \circ \beta$. We thus have a contradiction.
This proves the theorem.

6. A general sufficiency condition for the validity of the GG-scheme

The sufficiency condition of Theorem 3 does not seem to be polynomially testable in general. The only polynomially testable sufficieny conditions on the density matrix D, for which for any instance of the TSP, with cost matrix C and a permutation Γ, such that D is the density matrix of C^Γ, the GG-scheme with Γ as the starting permutation produces an optimal solution, are those reported in [2,4,5,11,15,16]. (See also [14].) In [4], it is proved that the non-negativity of D is a sufficient condition. From Equation 1, it follows that the Gilmore-Gomory case [7] is of this type. A minor generalization of the non-negative case is obtained in [5]. Both these cases can be easily shown to satisfy the condition of Theorem 3. The most generally known such polynomially testable sufficiency condition on D is the one reported in [2,15]. This condition is a special case of the condition of Theorem 3 and it properly generalizes the results in [4,5]. However, even this condition seems highly constrained. For example, consider Examples 1 & 2 of the previous section. Both examples satisfy the condition of Theorem 3. Hence the GG-scheme is valid for both cases. However, the density matrices shown in these two examples do not satisfy the polynomially testable sufficiency condition in [2,15].

We give below a more general special case of Theorem 3, which can be polynomially tested and which properly generalizes the polynomially testable sufficiency conditions in [2,15].

Definition 8. For a positive integer n and an $n \times n$ matrix A, and any $1 \leq i, j \leq q \leq u, v \leq n-1$ such that $|\{i, j, q, u, v\}| \geq 2$,

$$F_U^A(i, q, u, j, q, v) = M_{i,u,q,v}^A + M_{q,u,j,q-1}^A;$$

and

$$F_L^A(i, q, u, j, q, v) = M_{i,u,j,q}^A + M_{i,q,q+1,v}^A.$$

Lemma 5 *Suppose D is an $(n-1) \times (n-1)$ matrix satisfying the following two conditions:*
(i) $\forall$ $1 \leq i, j \leq q \leq u, v \leq n-1$ such that $|\{i, j, q, u, v\}| \geq 2$, except for the case $\{u = v = q$ and $i = j \neq q-1\}$, $F_U^D(i, q, u, j, q, v) \geq 0$; and
(ii) $\forall$ $1 \leq i, j \leq q \leq u, v \leq n-1$ such that $|\{i, j, q, u, v\}| \geq 2$, except for the case $\{i = j = q$ and $u = v \neq q+1\}$, $F_L^D(i, q, u, j, q, v) \geq 0$.

Let a circuit φ on N with its unique non-trivial subtour C and a set $\emptyset \neq S \subseteq N$ be such that each element of S lies in the range of C and every region of

C contains, at most, one element of S. Then, there exists a permutation ζ on N that is dense on S such that $D(\varphi) \geq D(\zeta)$.

Proof: Let the node set of C be X and let its range be $[a, b-1]$. Let $(b-a) = r$ and $|S| = m$. We prove the result by induction on r and m.
For $r = 1$, the result is obviously true.
Suppose, for some $k > 1$, the result is true $\forall\ r < k$ and $\forall\ 1 \leq m \leq r$. Let us now consider the case $r = k$.
Case 1: $X = \{a, b\}$: Let $S = \{x\}$ for some $a \leq x \leq b-1$.
If $k = 2$ then let us consider the case $x = a$. (The case $x = b-1$ follows similarly.) Let ω be the circuit with its unique non-trivial subtour on node set $\{a, b-1\}$. Then $D(\varphi) - D(\omega) = F_U^D(a, a+1, a+1, a, a+1, a+1) \geq 0$. If $k = 3$ and $x = a+1$, then let ω be the circuit with its unique non-trivial subtour on node set $\{a+1, a+2\}$. Then $D(\varphi) - D(\omega) = F_U^D(a, a+2, a+2, a+1, a+2, a+2) + F_L^D(a, a, a+2, a, a, a+1) \geq 0$ and in all other cases where $k \geq 3$, $x - a \geq 2$ or $b-1-x \geq 2$. Let us consider the case $x - a \geq 2$. (The other case follows similarly.) Let ω be the circuit with its unique non-trivial subtour on node set $\{x, b\}$. Then $D(\varphi) - D(\omega) = F_L^D(a, x-1, b-1, a, x-1, b-1) \geq 0$. In each of the above cases, the permutation ω has smaller value of r. Hence, the result follows by the induction hypothesis.
Case 2: $|X| > 2$: Suppose $a \notin S$. Let $u = \varphi^{-1}(a)$. If $(a+1) \in X$, then let $\omega = \varphi \circ \alpha_{a,u}$ and if $(a+1) \notin X$, then let $\omega = \varphi \circ \alpha_{a,u} \circ \alpha_{a+1,u}$. In either case, ω is a circuit with $(X \cup \{a+1\}) - \{a\}$ as the node set of its unique, nontrivial subtour C'. In the first case, $D(\varphi) - D(\omega) = F_U^D(a, a, u-1, a, a, \varphi(a) - 1) \geq 0$; and in the second case, $D(\varphi) - D(\omega) = F_L^D(a, a, u-1, a, a, \varphi(a) - 1) \geq 0$. In either case, the range of C' is $k-1$; and ω and S satisfy the conditions of the lemma. Hence, by the induction hypothesis, the result follows. The case when $(b-1) \notin S$ follows similarly.

Now, if $m = 1$, then either $a \notin S$ or $(b-1) \notin S$, and the result follows from the above.

Suppose the result is true $\forall\ m < t$, for some $k \geq t > 1$. Let us consider the case $m = t$.

If $t = k$, then φ is dense on S and the result follows trivially. If either $a \notin S$ or $(b-1) \notin S$, then the result follows as shown above. So let us consider the case when $\{a, b-1\} \subseteq S$, and $t < k$. Let $q = \min\{i : a < i < b-1; i \notin S\}$. We say that an arc $(i, j) \in E_\varphi$ crosses q if either $i \leq q < j$ or $j \leq q < i$.

Subcase (i): $q \notin X$: Let (u_1, v_1) and (v_2, u_2) be a pair of arcs crossing q encountered consecutively when we traverse the subtour C and such that $u_1 > q$. Let $\omega = \varphi \circ \alpha_{v_2,u_1} \circ \alpha_{v_2,q}$. Thus ω is obtained from φ by replacing subtour C by two subtours: C^1 on node set $X^1 \subseteq (X \cap \{i : i \leq q\}) \cup \{q\}$ and C^2 on node set $X^2 = X - X^1$; and $D(\varphi) - D(\omega) = F_U^D(v_2, q, u_1 - 1, v_1, q, u_2 - 1) \geq 0$.

Let $S^1 = S \cap X^1$ and $S^2 = S - S^1$. Let the ranges of X^1 and X^2 be $[a^1, q]$ and $[a^2, b]$ respectively. Then $S^1 \neq \emptyset$ and if $b - a^2 > 0$ then $S^2 \neq \emptyset$. Let ω^1 and ω^2 be circuits on N with unique non-trivial subtours C^1 and C^2, respectively. Thus, the pairs (ω^1, S^1) and (ω^2, S^2) satisfy the conditions of the lemma and $(q - a^1) < k$, $(b - a^2) \leq k$ and $|S^2| < t$. Hence, by the induction hypothesis, there exists a permutation ζ^1 dense on S^1 such that $D(\zeta^1) \leq D(\omega^1)$, and a permutation ζ^2 dense on S^2 such that $D(\zeta^2) \leq D(\omega^2)$. Let $\zeta = \zeta^1 \circ \zeta^2$. Then ζ is dense on S and $D(\zeta) = D(\zeta^1) + D(\zeta^2) \leq D(\omega^1) + D(\omega^2) \leq D(\varphi)$.

Subcase (ii): $q \in X$: Let (u_1, v_1) and (v_2, u_2) be a pair of arcs crossing q encountered consecutively when we traverse the subtour C such that (i) $u_1 > q$ and (ii) the directed path in G_φ from node v_1 to node v_2 contains the node q. If $(q+1) \in X$, then let $\omega = \varphi \circ \alpha_{v_2,u_1}$; else, let $\omega = \varphi \circ \alpha_{v_2,u_1} \circ \alpha_{q+1,u_1}$. Thus ω is obtained from φ by replacing the subtour C by two subtours: C^1 on node set say $X^1 \subseteq X \cap \{i : i \leq q\}$ and C^2 on node set $X^2 = (X \cup \{q+1\}) - X^1$.
In the first case,

$$D(\varphi) - D(\omega) = \begin{cases} F_U^D(v_2, v_1, u_1 - 1, v_1, v_1, u_2 - 1) \geq 0 & \text{if } v_2 \leq v_1, \\ F_U^D(v_2, v_2, u_1 - 1, v_1, v_2, u_2 - 1) \geq 0 & \text{otherwise.} \end{cases}$$

In the second case, $D(\varphi) - D(\omega) = F_L^D(v_2, q, u_1 - 1, v_1, q, u_2 - 1) \geq 0$.

Let $S^1 = S \cap X^1$ and $S^2 = S - S^1$. Let the ranges of X^1 and X^2 be $[a^1, q]$ and $[a^2, b]$ respectively. Let ω^1 and ω^2 be circuits on N with unique non-trivial subtours C^1 and C^2, respectively. If $q - a^1 > 0$, then $S^1 \neq \emptyset$. Also, $b - a^2 > 0$ and $S^2 \neq \emptyset$. Thus, the pairs (ω^1, S^1) and (ω^2, S^2) satisfy the conditions of the lemma. Since $(q - a^1) < k$, it follows by the induction hypothesis that there exists a permutation ζ^1 dense on S^1 such that $D(\zeta^1) \leq D(\omega^1)$. Also, $(b - a^2) \leq k$ and $|S^2| \leq t$. If $|S^2| < t$, then by induction hypothesis, there exists a permutation ζ^2, dense on S^2 such that $D(\zeta^2) \leq D(\omega^2)$. If $|S^2| = t$, then $(b - a^2) = k$ and $q \notin X^2$. Hence, by Subcase (i) above, there exists a permutation ζ^2 dense on S^2 such that $D(\zeta^2) \leq D(\omega^2)$. In either case, let $\zeta = \zeta^1 \circ \zeta^2$. Then ζ is dense on S and

$D(\zeta) = D(\zeta^1) + D(\zeta^2) \leq D(\omega^1) + D(\omega^2) \leq D(\varphi)$. This proves the lemma.

Lemma 6 *Suppose D is an $(n-1) \times (n-1)$ matrix satisfying the following three conditions:*
(i) $\forall\ 1 \leq i,j \leq q \leq u,v \leq n-1$ such that $|\{i,j,q,u,v\}| \geq 2$, except for the case $\{u = v = q$ and $i = j \neq q-1\}$, $F_U^D(i,q,u,j,q,v) \geq 0$
(ii) $\forall\ 1 \leq i,j \leq q \leq u,v \leq n-1$ such that $|\{i,j,q,u,v\}| \geq 2$, except for the case $\{i = j = q$ and $u = v \neq q+1\}$, $F_L^D(i,q,u,j,q,v) \geq 0$;
(iii) for any $1 \leq i \leq n-2$, $D[\{i,i+1\}] \geq 0$.

Let φ be the circuit on N with a unique non-trivial subtour C. Suppose the node set X of C is not of the type $\{i,i+1\}$. Then $D(\varphi) \geq 0$.

Proof: We prove the result by induction on the size $|X| = r$ of the subtour C.
If $X = \{i,j\}$ with $|i-j| > 1$, then
$D(\varphi) = F_U^D(i,i,j-1,i,i,j-1) \geq 0$.

If $X = \{i,j,k\}$ with $i < j < k$, and if $(j-i) = (k-j) = 1$, the result follows from the condition (iii) of the lemma.
Otherwise, if $j > i+1$, then let $\varphi(i) = x$ and $\varphi^{-1}(i) = y$. Let $\omega = \varphi \circ \alpha_{i,y} \circ \alpha_{i+1,y}$.

The node set of the unique subtour C' of ω is $\{i+1,j,k\}$ and
$D(\varphi) - D(\omega) = F_L^D(i,i,y-1,i,i,x-1) \geq 0$.
If $j = i+1$, then we must have $k > j+1$. Let $\varphi(k) = x$ and $\varphi^{-1}(k) = y$; and let $\omega = \varphi \circ \alpha_{y,k} \circ \alpha_{y,k-1}$. The node set of the unique subtour C' of ω is $\{i,j,k-1\}$ and $D(\varphi) - D(\omega) = F_U^D(y,k-1,k-1,x,k-1,k-1) \geq 0$.

In either of the above cases, by repeating this argument with ω, we end up with a circuit ζ such that $D(\varphi) \geq D(\zeta)$ and the node set of the unique subtour $\bar{C}$ of ζ is of the form $\{\ell,\ell+1,\ell+2\}$, which implies, by condition (iii) of the lemma, that $D(\zeta) \geq 0$.

Now suppose the result is true for all $r < k$ and for some $k > 3$. Let us consider the case $r = k$. Let $X = \{i_1,i_2,\cdots,i_k\}$ where $i_1 < i_2 < \cdots < i_k$. Let $\varphi(i_1) = i_u$ and $\varphi^{-1}(i_1) = i_v$. Let $\omega = \varphi \circ \alpha_{i_v,i_1}$. The node set of the unique subtour C' of ω is $\{i_2,\cdots,i_k\}$; and

$D(\varphi) - D(\omega) = F_U^D(i_1,i_1,i_v-1,i_1,i_1,i_u-1) \geq 0$.
Thus, by the induction hypothesis, we get, $D(\varphi) \geq D(\omega) \geq 0$.

Lemma 7 *Suppose D is an $(n-1) \times (n-1)$ matrix satisfying the following three conditions:*
(i) $\forall\ 1 \leq i,j \leq q \leq u,v \leq n-1$ such that $|\{i,j,q,u,v\}| \geq 2$, except for the case $\{u = v = q$ and $i = j \neq q-1\}$, $F_U^D(i,q,u,j,q,v) \geq 0$;
(ii) $\forall\ 1 \leq i,j \leq q \leq u,v \leq n-1$ such that $|\{i,j,q,u,v\}| \geq 2$, except for the case $\{i = j = q$ and $u = v \neq q+1\}$, $F_L^D(i,q,u,j,q,v) \geq 0$;
(iii) for any $1 \leq i \leq n-2$, $D[\{i,i+1\}] \geq 0$.

Let φ be an arbitrary circuit. Let the node set and range of its unique subtour C be X and $[p,m-1]$, respectively. Let $S \subseteq N$ be such that each element of S lies in the range of C; every region of C contains, at most, one element of S; and there exists a region $[a,b-1]$ of C that contains no element of S. Then there exists $p \leq i \leq a \leq q \leq b-1 \leq j \leq m-1$, and a permutation ζ that is dense on S such that
$D(\varphi) \geq D(\zeta) + M_{i,j,q,q}^D$, *or* $D(\varphi) \geq D(\zeta) + M_{q,q,i,j}^D$.

Proof: Let (s_1,t_1) and (t_2,s_2) be a pair of arcs crossing node a that are encountered consecutively when we traverse the subtour C, such that (i) $s_1 \geq b$ and (ii) the directed path in G_φ from node t_1 to node t_2 contains the node a. Let $\omega = \varphi \circ \alpha_{t_2,s_1}$. Thus ω is obtained from φ by replacing subtour C with two subtours: C^1 on node set $X^1 \subseteq X \cap \{i : i \leq a\}$ and C^2 on node set $X^2 = X - X^1$. It is readily seen that

$$D(\varphi) - D(\omega) = \begin{cases} F_U^D(t_2,t_1,s_1-1,t_1,t_1,s_2-1) \geq 0 \text{ if } t_2 \leq t_1, \\ F_U^D(t_2,t_2,s_1-1,t_1,t_2,s_2-1) \geq 0 \text{ otherwise.} \end{cases}$$

In either case, $D(\varphi) - D(\omega) \geq M_{i,j,q,q}^D$ or $M_{q,q,i,j}^D$ for some $p \leq i \leq a \leq q \leq b-1 \leq j \leq v-1$.

Let ω^1 and ω^2 be the circuits on N with unique subtours C^1 and C^2, respectively.
Let $X^1 = \{i_1,i_2,\ldots,i_\ell = a\}$; let $S^1 = \{x_j : x_j$ is the unique element of S in the region of X having lower limit $i_j; 1 \leq j < \ell\}$; and let $S^2 = S - S^1$. Then, every element of S^1 (S^2) lies in the range of X^1 (X^2) and every region of X^1 (X^2) contains, at most, one element of S^1 (S^2). Hence, by Lemma 5, there exist permutations ζ^1 and ζ^2 dense, respectively, on S^1 and S^2, such that $D(\omega^1) \geq D(\zeta^1)$ and $D(\omega^2) \geq D(\zeta^2)$. Let $\zeta = \zeta^1 \circ \zeta^2$. Then ζ is dense on S and $D(\zeta) = D(\zeta^1) + D(\zeta^2) \leq D(\omega^1) + D(\omega^2) \leq D(\varphi) - M_{i,j,q,q}^D$ or $D(\varphi) - M_{q,q,i,j}^D$ for some $p \leq i \leq a \leq q \leq b-1 \leq j \leq m-1$. This proves the lemma.

Theorem 8 *Suppose D is an $(n-1) \times (n-1)$ matrix satisfying the following five conditions:*
(i) $\forall\ 1 \leq i,j \leq q \leq u,v \leq n-1$ such that

$|\{i,j,q,u,v\}| \geq 2$, *except for the case* $\{u = v = q$ *and* $i = j \neq q-1\}$, $F_U^D(i,q,u,j,q,v) \geq 0$.
(ii) $\forall\ 1 \leq i,j \leq q \leq u,v \leq n-1$ *such that* $|\{i,j,q,u,v\}| \geq 2$, *except for the case* $\{i = j = q$ *and* $u = v \neq q+1\}$, $F_L^D(i,q,u,j,q,v) \geq 0$.
(iii) for any $1 \leq i \leq n-2$, $D[\{i,i+1\}] \geq 0$.
(iv) For any $1 \leq i \leq u \leq j < n$ *and any* $1 \leq k < i$ *or* $j < k < n$, $M^D_{i,j,u,u} + d_{kk} \geq 0$ *and* $M^D_{u,u,i,j} + d_{k,k} \geq 0$.
(v) For any principal submatrix D' of D on a consecutive subset of its rows/columns, the corresponding instance of the TSP is pyramidally solvable.

Then, for any cost matrix C and any permutation Γ on N, such that D is the density matrix of C^Γ, the GG-scheme with Γ as the suitable starting permutation produces an optimal solution to the corresponding TSP.

Proof: Let D be a matrix satisfying the conditions of the theorem. We shall show that it satisfies the conditions of Theorem 3.

Thus let Ψ be any arbitrary permutation on N. Suppose Ψ has ℓ non-trivial subtours and G^I_Ψ has r connected components of sizes $\ell_1, \ell_2, \ldots, \ell_r$. Let $X^1_\Psi, X^2_\Psi, \ldots, X^r_\Psi$ be the unions of the node sets of the non-trivial subtours of Ψ corresponding, respectively, to the node sets of the r connected components of G^I_Ψ. Let S be any subset of N with a partition $\{S_1, S_2, \ldots, S_r\}$ such that (i) for any $1 \leq i \leq r$, every element of S_i lies in the range X^i_Ψ; every region of X^i_Ψ contains, at most, one element of S_i; and where $|S_i| \leq (|X^i_\Psi| - \ell_i)$; and where (ii) $|S| \equiv ((\sum_{i=1}^r |X^i_\Psi|) - \ell) \text{mod} 2$.

We shall, first of all, produce a permutation Ψ^0 on N with $D(\Psi^0) \leq D(\Psi)$, such that Ψ^0 has precisely r non-trivial subtours, one on each of the node sets $X^1_\Psi, X^2_\Psi, \ldots, X^r_\Psi$. If $\ell = r$, then $\Psi^0 = \Psi$. Else, let C^1 and C^2 be two non-trivial subtours of Ψ such that their ranges, $[i_1, j_1]$ and $[i_2, j_2]$, intersect. Without loss of generality, let us assume that $i_1 < i_2 < j_1$. Then there exist nodes u and v of subtours C^1 and C^2, respectively, such that $i_1 \leq u < i_2 < v$ and $\Psi(u) > i_2$ and $\Psi(v) = i_2$. Let $\Psi' = \Psi \circ \alpha_{u,v}$. In Ψ', the two subtours C^1 and C^2 of Ψ are combined into one, while the other subtours of Ψ' are precisely those of Ψ. Furthermore, $D(\Psi) - D(\Psi') = M^D_{u,v-1,i_2,\Psi(u)-1} = F^D_U(u, i_2, v-1, i_2, i_2, \Psi(u)-1) \geq 0$. By repeating the process, we get the desired permutation Ψ^0.

Ψ^0 has r non-trivial subtours, say $C_1, C_2, \ldots, C_r$, on the node sets $X^1_\Psi, X^2_\Psi, \ldots, X^r_\Psi$, respectively. For each $i \in \{1, 2, \ldots, r\}$, let ζ^i be the circuit on N with C_i as its unique non-trivial subtour. Let $Y \subseteq \{1, 2, \ldots, r\}$ be such that $\forall\ j \in Y$, X^j_Ψ is of the type $\{i, i+1\}$ and $S_j = \emptyset$. Let $|Y| = k$.

Case (i) $k = 0$: In this case, $D(\Psi) \geq D(\Psi^0) = \sum_{i=1}^r D(\zeta^i) \geq \sum_{i=1}^r D[S_i]$, (by Lemma 5 and condition (v) of the Theorem) $= D[S]$.

Case (ii) $k = 1$: Let C_1 be the only subtour with a node set of the type $\{i, i+1\}$ and $S_1 = \emptyset$. In this case, since $|S| \equiv ((\sum_{i=1}^r |X^i_\Psi|) - \ell) \text{mod} 2$ and $\forall\ 1 \leq i \leq r$, $|S_i| \leq (|X^i_\Psi| - \ell_i)$, there exists some $1 < j \leq r$, such that the set X^j_Ψ is not of the type $\{i, i+1\}$ and some region of X^j_Ψ contains no element of S_j. Let the range of X^j_Ψ be $[a, b-1]$. Then, by Lemma 7, there exists a permutation ζ that is dense on S_j and some $i \leq a \leq u \leq b-1 \leq j$ such that $D(\zeta^j) \geq D(\zeta) + M^D_{i,j,u,u}$ or $D(\zeta^j) \geq D(\zeta) + M^D_{u,u,i,j}$. Delete subtours C_1, and C_j from Ψ^0 and add to them the non-trivial subtours of ζ to get a new permutation Ψ^1. Then, by condition (iv) of the theorem,

$$D(\Psi^0) - D(\Psi^1) = D(\zeta^1) + D(\zeta^j) - D(\zeta)$$
$$\geq \begin{cases} D(\zeta^1) + M^D_{i,j,u,u} \geq 0 \\ \quad\quad or \\ D(\zeta^1) + M^D_{u,u,i,j} \geq 0 \end{cases}$$

Now, $D(\Psi^1) = D(\zeta) + \sum\{D(\zeta^i) : i \in \{2, 3, \ldots, r\}; i \neq j\}$. By condition (v) of the theorem, $D(\zeta) \geq D[S_j]$ and by Lemma 5 and condition (v) of the theorem, $D(\zeta^i) \geq D[S_i]\ \forall\ 2 \leq i \leq r,\ i \neq j$. Hence, $D(\Psi^1) \geq D[S]$.

Case (iii) $k > 1$ and even: Let $Y = \{1, 2, \ldots, k\}$ and let Ψ^1 be the permutation with non-trivial subtours $\{C_i : i \in \{k+1, k+2, \ldots, r\}\}$. Then by condition (iv) of the theorem, $D(\zeta^i) + D(\zeta^{i+1}) \geq 0\ \forall i = 1, 3, \ldots, k-1\}$; and hence, $D(\Psi^1) \leq D(\Psi^0)$. By Lemma 5 and condition (v) of the theorem, $D(\Psi^1) = \sum\{D(\zeta^i) : i \in \{k+1, k+2, \ldots, r\}\} \geq \sum\{D[S_i] : i \in \{k+1, k+2, \ldots, r\}\} = D[S]$.

Case (iv) $k > 1$ and odd: Let $Y = \{1, 2, \ldots, k\}$ and let Ψ^1 be the permutation with non-trivial subtours $\{C_i : i \in \{k, k+2, \ldots, r\}\}$. By condition (iv) of the theorem, $D(\zeta^i) + D(\zeta^{i+1}) \geq 0\ \forall i = 1, 3, \ldots, k-2\}$; and hence, $D(\Psi^1) \leq D(\Psi^0)$. By Case (ii) above, $D(\Psi^1) \geq D[S]$.

Thus, the matrix D satisfies the sufficiency condition of Theorem 3. The result now follows from Theorem 3.

Examples 1 and 2 in Section 5 satisfy the conditions of Theorem 8. Thus, the theorem properly generalizes the sufficiency conditions in [2,15].

Conditions (i), (ii), (iii) and (iv) of Theorem 8 can be trivially tested in $O(n^4)$ time. Testing, in general, whether for a given density matrix D the corresponding instance of the TSP is pyramidally solvable, is Co-NP-hard [14]. However, we do not know if a polynomial testing scheme exists if the matrix also satisfies condition (i)-(iv) of the theorem. In general, we can replace condition (v) of the Theorem by any one of the polynomially testable sufficiency conditions for pyramidal solvability of the TSP [14] to get a general, polynomially testable sufficiency condition for the validity of the GG-scheme.

7. Conclusion

One of the most well-known polynomially solvable cases of the TSP is the Gilmore-Gomory TSP. Several researchers have studied and generalized the subtour patching scheme of Gilmore and Gomory and developed polynomially testable sufficiency conditions for a TSP to be polynomially solvable. However, finding a good characterization of the class of the TSP, for which the GG-scheme produces an optimal solution, is still an important open problem. In this paper, we have given some necessary conditions and a new polynomially testable sufficiency condition for the validity of the GG-scheme that properly includes all the previously known conditions.

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