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Vertex 3-colorability of claw-free graphs

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Abstract

*The 3-*COLORABILITY *problem is NP-complete in the class of claw-free graphs. In this paper we study the computational complexity of the problem in subclasses of claw-free graphs defined by forbidding finitely many additional subgraphs (line graphs and claw-free graphs of bounded vertex degree being examples of such classes). We prove a necessary condition for the polynomial-time solvability of the problem in such classes and propose a linear-time solution for an infinitely increasing hierarchy of classes that meet the condition. To develop such a solution for the basis of this hierarchy, we generalize the notion of locally connected graphs that has been recently studied in the context of the 3-*COLORABILITY *problem.*

Key words: Vertex 3-colorability; Claw-free graphs; Polynomial-time algorithm

1. Introduction

A 3-coloring of a graph G is a mapping that assigns a color from set $\{0, 1, 2\}$ to each vertex of G in such a way that any two adjacent vertices receive different colors. The 3-COLORABILITY problem is that of determining if there exists a 3-coloring of a given graph, and if so, finding it. Alternatively, one can view a 3-coloring of a graph as a partitioning its vertices into three independent sets, called *color classes*. If such a partition is unique, the graph is said to have a *unique coloring*.

From an algorithmic point of view, 3-COLORABILITY is a difficult problem, i.e., it is NP-complete. Moreover, the problem remains difficult even under substantial restrictions, for instance, for graphs of vertex degree at most four [6]. On the other hand, for graphs in some special classes, such as locally connected graphs [7], the problem can be solved efficiently, i.e., in polynomial time. Recently, a number of papers investigated computational complexity of the problem on graph classes defined by forbidden induced subgraphs (see e.g. [11–15]). In the present paper, we study this problem restricted to claw-free graphs, the class which lately received considerable attention in the literature [1,3,8]. The 3-COLORABILITY problem in claw-free graphs includes, as a subproblem, EDGE 3-COLORABILITY of general graphs, i.e., the problem of determining whether the edges of a given graph can be assigned colors from set $\{0, 1, 2\}$ so that any two edges sharing a vertex receive different colors. Indeed, by associating with a graph G its line graph $L(G)$ (i.e., the graph with $V(L(G)) = E(G)$ and two vertices being adjacent in $L(G)$ if and only if the respective edges of G have a vertex in common), one can transform the question of edge 3-colorability of G into the question of vertex 3-colorability of $L(G)$. In conjunction with the NP-completeness of EDGE 3-COLORABILITY [5], this implies the NP-completeness of (vertex) 3-COLORABILITY of line graphs. It is known that every line graph is claw-free. Moreover, the line graphs constitute a *proper* subclass of claw-free graphs, which can be characterized by 8 additional forbidden induced subgraphs (see e.g. [4] for the complete list of minimal non-line graphs). In this paper we study computational complexity of the problem in other subclasses of claw-free graphs defined by finitely many forbidden induced subgraphs. First, we prove a necessary condition for polynomial-time solvability of the problem in such classes, and then for an infinitely increasing hierarchy of classes that meet the condition, we propose a linear-time solution. To develop such a solution for the basis of this hierarchy, we generalize the notion of locally connected graphs that has been recently studied in the context of the 3-COLORABILITY problem.

All graphs in this paper are finite, loopless and without multiple edges. The vertex set of a graph G is denoted $V(G)$ and the edge set $E(G)$. A subgraph of G is called *induced* by a set of vertices $A \subset V$ if it can

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be obtained from G by deleting the vertices outside A. We denote such a subgraph by $G[A]$. If G contains no induced subgraphs isomorphic to a graph in a set M, we say that G is M-free. The *neighborhood* of a vertex v, denoted by $N(v)$, is the set of all vertices adjacent to v. The number of neighbors of a vertex v is called its *degree* and is denoted $\deg(v)$. $\Delta(G)$ is the maximum degree and $\delta(G)$ is the minimum degree of a vertex in a graph G. If $\Delta(G) = \delta(G)$, the graph G is called *regular of degree* $\Delta(G)$. In particular, regular graphs of degree 3 are called *cubic*. The *closed neighborhood* of v is the set $N[v] := N(v) \cup \{v\}$.

As usual, P_n, C_n and K_n stand, respectively, for a chordless path, a chordless cycle and a complete graph on n vertices. $K_{n,m}$ is the complete bipartite graph with parts of size n and m. A wheel W_n is obtained from a cycle C_n by adding a dominating vertex, i.e., a vertex adjacent to every vertex of the cycle. For some particular graphs we use special names: $K_{1,3}$ is a *claw*, $K_4 - e$ (i.e., the graph obtained from K_4 by removing one edge) is a *diamond*, while a gem is the graph obtained from a P_4 by adding a dominating vertex.

Notice that in the context of 3-colorability it is enough to consider connected graphs only, since if a graph is disconnected, then the problem can be solved for each of its connected components separately. Therefore, without loss of generality, we assume all graphs in this paper are connected.

We can also assume that $\delta(G) \geq 3$. Indeed, if v is a vertex of G of degree less than three, then G has a 3-coloring if and only if the graph obtained from G by deleting v has one. Moreover, whenever we deal with claw-free graphs, we can restrict ourselves to graphs of vertex degree at most four. Indeed, it is not difficult to verify that every graph with five vertices contains either a triangle or its complement or a C_5. Therefore, every graph with a vertex of degree five or more contains either a $claw$ or K_4 or W_5. Since K_4 and W_5 are not 3-colorable, we conclude that every $claw$-free graph, which is 3-colorable, has maximum vertex degree at most 4.

2. NP-completeness

In this section we establish several results on the NP-completeness of the 3-COLORABILITY problem in subclasses of $claw$-free graphs. We start by recalling the following known fact.

Lemma 1 *The 3-*COLORABILITY *problem on* $(claw, diamond)$*-free graphs of maximum vertex degree at most four is NP-complete.*

The result follows by a reduction from EDGE 3-COLORABILITY of C_3-free cubic graphs, which is an NP-complete problem [5]. It is not difficult to verify that if G is a C_3-free cubic graph, then $L(G)$ is a $(claw, diamond, K_4)$-free regular graph of degree four. It is interesting to note that the inverse reduction is also valid: with any $(claw, diamond, K_4)$-free 4-regular graph H, one can associate a C_3-free cubic graph G such that $L(G) = H$. Indeed, each maximal clique in H (which must be of size 3) becomes a vertex of G, and two vertices are adjacent in G if the respecrtive cliques of H share a vertex.

To establish more results, let us introduce more definitions and notations. First, we introduce the following three operations:

- replacement of an edge by a diamond (Figure 1);

Fig. 1. Replacement of an edge by a diamond

- implantation of a diamond at a vertex (Figure 2);

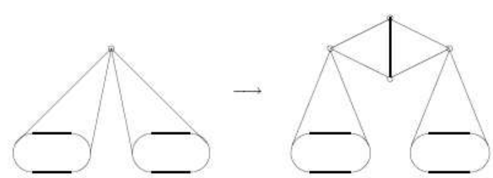

Fig. 2. Diamond implantation

- implantation of a triangle into a triangle (Figure 3)

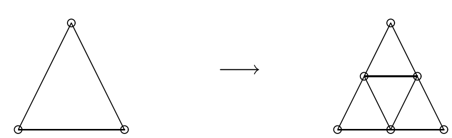

Fig. 3. Triangle implantation

Observe that a graph obtained from a graph G by diamond or triangle implantation is 3-colorable if and only if G is.

Denote by

$T_{i,j,k}$ - the graph represented in Figure 4;
$T^1_{i,j,k}$ - the graph obtained from $T_{i,j,k}$ by replacing each edge, which is not in the central triangle, by a diamond; and
$T^2_{i,j,k}$ - the graph obtained from $T^1_{i,j,k}$ by implanting into its central triangle a new triangle.

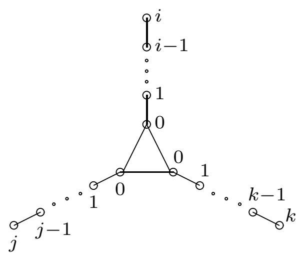

Fig. 4. The graph $T_{i,j,k}$

Finally, denote by

$\mathcal{T}$ - the class of graphs every connected component of which is an induced subgraph of a graph of the form $T_{i,j,k}$;
$\mathcal{T}^1$ - the class of graphs every connected component of which is an induced subgraph of a graph of the from $T^1_{i,j,k}$; and
$\mathcal{T}^2$ - the class of graphs every connected component of which is an induced subgraph of a graph of the from $T^2_{i,j,k}$.

Notice that none of the classes $\mathcal{T}^1$ and $\mathcal{T}^2$ contains the other. Indeed, $\mathcal{T}^2 \setminus \mathcal{T}^1$ contains a gem, while $\mathcal{T}^1 \setminus \mathcal{T}^2$ contains the graph $T^{\Delta}_{0,0,0}$ (see Figure 5 for the definition of $T^{\Delta}_{i,j,k}$).

Theorem 2 *Let X be a subclass of $claw$-free graphs defined by a finite set M of forbidden induced subgraphs. If $M \cap \mathcal{T}^1 = \emptyset$ or $M \cap \mathcal{T}^2 = \emptyset$, then the 3-*COLORABILITY *problem is NP-complete for graphs in the class X.*

Proof: We prove the theorem by a reduction from the class of $(claw, gem, W_4)$-free graphs of vertex degree at most 4, where the problem is NP-complete, since this class is an extension of $(claw, diamond)$-free graphs of maximum degree 4.

Let G be a $(claw, gem, W_4)$-free graph of vertex degree at most 4. Without loss of generality we can also assume that G is K_4-free (since otherwise G is not 3-colorable) and every vertex of G has degree at least 3. We will show that if $M \cap \mathcal{T}^1 = \emptyset$ or $M \cap \mathcal{T}^2 = \emptyset$, then G can be transformed in polynomial time into a graph in X, which is 3-colorable if and only if G is. Assume first that $M \cap \mathcal{T}^1 = \emptyset$.

Let us call a triangle in G *private* if it is not contained in any diamond. Also, we shall call a vertex x *splittable* if the neighborhood of x can be partitioned into two disjoint cliques X_1, X_2 with no edges between them. In particular, in a $(claw, gem, W_4, K_4)$-free graph of degree at most 4 every vertex of a private triangle is splittable. Also, it is not difficult to verify that in such a graph every chordless cycle of length at least 4 contains a splittable vertex.

Given a splittable vertex x with cliques X_1, X_2 in its neighborhood, apply the diamond implantation k times, i.e., replace x with two new vertices x_1 and x_2, connect x_i to every vertex in X_i for $i = 1, 2$, and connect x_1 to x_2 be a chain of k diamonds. Obviously the graph obtained in this way is 3-colorable if and only if G is. We apply this operation to every splittable vertex of G and denote the resulting graph by $G(k)$. Observe that $G(k)$ is $(C_4, \ldots, C_k)$-free and the distance between any two private triangles is at least k.

Let us show that if k is larger than the size of any graph in M, then $G(k)$ belongs to X. Assume by contradiction that $G(k)$ does not belong to X, then it must contain an induced subgraph $A \in M$. We know that A cannot contain chordless cycles $C_4, \ldots, C_k$. Moreover, it cannot contain cycles of length greater than k, since A has at most k vertices. For the same reason, each connected component of A contains at most one private triangle. But then $A \in \mathcal{T}^1$, contradicting our assumption that $M \cap \mathcal{T}^1 = \emptyset$.

Now assume that $M \cap \mathcal{T}^2 = \emptyset$. In this case, we first transform G into $G(k)$ and then implant a new triangle into each private triangle of G. By analogy with the above case, we conclude that the graph obtained in this way belongs to X, thus completing the necessary reduction. □

The above theorem not only proves the NP-completeness of the problem in certain graph classes, but also suggests what classes can have the potential for accepting a polynomial-time solution. In particular, for subclasses of claw-free graphs defined by a single additional induced subgraph we obtain the following corollary of Theorem 2 and Lemma 1.

Corollary 3 *If X is the class of $(claw, H)$-free graphs, then the 3-*COLORABILITY *problem can be solved in polynomial time in X only if $H \in \mathcal{T}^1 \cap \mathcal{T}^2$.*

The intersection $\mathcal{T}^1 \cap \mathcal{T}^2$ includes, for instance, the class $\mathcal{T}$. More generally, it includes any graph every connected component of which is an induced subgraph of a graph of the form $T^{\Delta}_{i,j,k}$ with at least two non-zero indices (see Figure 5). In the next section, we analyze some of $(claw, H)$-free graphs with $H \in \mathcal{T}^1 \cap \mathcal{T}^2$ and derive polynomial-time solutions for them.

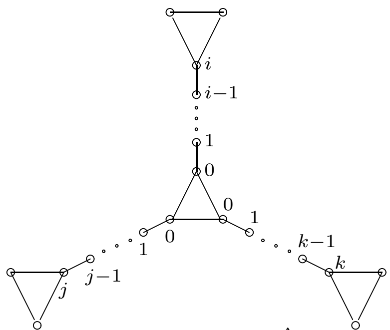

Fig. 5. The graph $T^{\Delta}_{i,j,k}$

3. Polynomial-time results

We start by reporting several results that are known from the literature. It has been shown in [13] that the 3-COLORABILITY problem can be solved in polynomial time for $(claw, T_{0,0,2})$-free graphs. More generally, we can show that

Theorem 4 *The 3-COLORABILITY problem can be solved in polynomial time in the class of $(claw, T_{i,j,k})$-free graphs for any i, j, k.*

Proof: First, observe that 3-COLORABILITY is a problem solvable in polynomial time on graphs of bounded clique-width [2]. From the results in [10] it follows that $(claw, T_{i,j,k})$-free graphs of bounded vertex degree have bounded clique-width for any i, j, k. In the introduction, it was observed that $claw$-free graphs containing a vertex of degree 5 or more are not 3-colorable, which provides the desired conclusion. □

Another solvable case described in [13] deals with $(claw, hourglass)$-free graphs, where an hourglass is the graph consisting of a vertex of degree 4 and a couple of disjoint edges in its neighborhood. Unfortunately, the authors of [13] do not claim any time complexity for their solution. In Section 3.1. we show that this case can be solved in linear time by generalizing the notion of locally connected graphs studied recently in connection with the 3-COLORABILITY problem. Then in Section 3.2. we extend this result to an infinitely increasing hierarchy of subclasses of claw-free graphs.

3.1. *Almost locally connected graphs*

A graph G is *locally connected* if for every vertex $v \in V$, the graph $G[N(v)]$ induced by the neighborhood of v is connected. The class of locally connected graphs has been studied in [7] in the context of the 3-COLORABILITY problem.

A notion, which is closely related to 3-coloring, is 3-*clique ordering*. In a connected graph G, an ordering $(v_1, \ldots, v_n)$ of vertices is called a 3-*clique ordering* if v_2 is adjacent to v_1, and for each $i = 3, \ldots, n$, the vertex v_i forms a triangle with two vertices preceding v_i in the ordering.

It is not difficult to see that for a graph with a 3-clique-ordering, the 3-COLORABILITY problem is solvable in time linear in the number of edges. (We will refer to such running time of an algorithm as *linear time*.) In [7], it has been proved that if G is connected and locally connected, then G admits a 3-clique-ordering and it can be found in linear time. Therefore, the 3-COLORABILITY problem in the class of locally connected graphs can be solved in linear time.

Now we introduce a slightly broader class and show that the 3-colorability of graphs of vertex degree at most 4 in the new class can be decided in linear time. This will imply, in particular, a linear-time solution for $(claw, hourglass)$-free graphs.

We say that a graph G is *almost locally connected* if the neighborhood of each vertex either induces a connected graph or is isomorphic to $K_1 \cup K_2$ (disjoint union of an edge and a vertex). In other words, the neighborhoods of all vertices are connected, or, if the degree of a vertex is 3, we allow the neighborhood to be disconnected, provided it consists of two connected components, one of which is a single vertex. If v is a vertex of degree 3 and w is an isolated vertex in the neighborhood of v, then we call the edge vw a *pendant edge*.

A maximal (with respect to set inclusion) subset of vertices that induces a 3-clique orderable graph will be called *3-clique orderable component*. Since a pendant edge belongs to no triangle in an almost locally connected graph, the endpoints of the pendant edge form a 3-clique orderable component, in which case we call it *trivial*. For the non-trivial 3-clique orderable components we prove the following helpful claim.

Claim 5 *Let G be an almost locally connected graph* *with $\Delta(G) \leq 4$ and with at least 3 vertices.*

(1) Every non-trivial 3-clique orderable component of G contains at least 3 vertices.

(2) Any two non-trivial 3-clique orderable components are disjoint.

(3) Any edge of G connecting two vertices in different non-trivial 3-clique orderable components is pendant.

Proof: To see (1), observe that in an almost locally connected graph with at least 3 vertices every non-pendant edge belongs to a triangle.

To prove (2), suppose G contains two non-trivial 3-clique orderable components M_1 and M_2 sharing a vertex v. Without loss of generality let v have a neighbor w in $M_1 - M_2$. Since M_1 is non-trivial, the edge vw cannot be pendant, and hence there is a vertex $u \in M_1$ adjacent both to v and w. Notice that u cannot belong to M_2, since otherwise $M_2 \cup \{w\}$ induces a 3-clique orderable graph contradicting maximality of M_2. On the other hand, M_2 must have a triangle containing vertex v, say v, x, y. If neither u nor w has a neighbor in $\{x, y\}$, then G is not locally connected. If there is an edge between $\{u, w\}$ and $\{x, y\}$, then M_2 is not a maximal set inducing a 3-clique orderable graph. This contradiction proves (2), which in its turn implies (3) in an obvious way. □

A natural corollary from the above claim is that an almost locally connected graph admits a unique partition into 3-clique orderable components and such a partition can be found in linear time.

Now we present a linear-time algorithm that, given a connected, almost locally connected graph G, decides 3-colorability of G and finds a 3-coloring, if one exists. In the algorithm, the operation of *contraction* of a set of vertices A means substitution of A by a new vertex adjacent to every neighbor of the set A.

ALGORITHM $\mathcal{A}$

Step 1. Find the unique partition $V_1, \ldots, V_k$ of G into 3-clique orderable components.

Step 2. If one of the graphs $G[V_i]$ is not 3-colorable, return the answer 'G IS NOT 3-COLORABLE'. Otherwise, 3-color each 3-clique orderable component.

Step 3. Create an auxiliary graph H, contracting color classes in each 3-clique orderable component V_i and then removing vertices of degree 2.

Step 4. If H is isomorphic to K_4, return the answer 'G IS NOT 3-COLORABLE'. Otherwise, 3-color H.

Step 5. Expand the coloring of H to a coloring of G and return it.

To show that the algorithm is correct and runs in linear time, we need to prove two properties of the auxiliary graph built in Step 3 of the algorithm.

Claim 6 *Let H be the auxiliary graph built in Step 3 of the algorithm. G is 3-colorable if and only if H is.*

Proof: If G is 3-colorable, then each 3-clique orderable component has a unique coloring. Therefore, the graph obtained by contracting color classes of each 3-clique orderable component (and, possibly, removing vertices of degree 2) is 3-colorable.

If H is 3-colorable, then vertices of degree 2 that were deleted in Step 3 can be restored and receive colors that are not assigned to their neighbors. Let a be a vertex of a triangle in H corresponding to a 3-clique orderable component in G. The edges that a is incident to (but not the triangle edges) correspond to the pendant edges of this component. Each is incident to two vertices in different color classes in H and will be in G. □

From the definition of 3-clique-ordering one can derive the following simple observation.

Claim 7 *If G has a 3-clique-ordering, then $|E(G)| \geq 2|V(G)| - 3$.*

Claim 8 *Let G be a connected and almost locally connected graph of maximum vertex degree at most 4, and H be the auxiliary graph built in Step 3 of the algorithm. Then, $\Delta(H) \leq 3$.*

Proof: Let T be a 3-clique orderable component of G. Denote by n_2, n_3, n_4 the number of vertices of degree $2, 3$ and 4 in $G[T]$, respectively. Also, let m' be the number of edges in $G[T]$. Then, $2m' = 2n_2 + 3n_3 + 4n_4$. In addition, $m' \geq 2(n_2 + n_3 + n_4) - 3$ (by Claim 7). Therefore, $n_2 \leq 3$. Notice that any pendant edge in G is adjacent to two vertices of degree 3 that are of degree 2 in the graphs induced by their 3-clique orderable components. Therefore, the number of pendant edges of a 3-clique orderable component is at most 3.

If a, b, c are vertices of a triangle corresponding to a locally connected component, and three pendant edges are incident to a, then b and c are removed in Step 4 of the algorithm, as both have degree 2, and a has degree 3 in H. If a is incident to two pendant edges, and b to one, then c is removed (as it has degree 2) and both b and c have degree at most 3 in H. If each of the vertices a, b, c is incident to one pendant edge only, then the degree of each of them is at most 3. □

Theorem 9 *Algorithm $\mathcal{A}$ decides 3-colorability of a connected, almost locally connected graph G with maximum vertex degree at most 4 and finds a 3-coloring of G, if one exists, in linear time.*

Proof: The correctness of the algorithm follows from Claims 6, 8 and the observation that by the Brooks' theorem the only graph of maximum degree 3 that is not 3-colorable is K_4.

The first step of the algorithm can be implemented as a breadth first search and performed in linear time. 3-coloring each of the locally connected components in linear time can be done by applying the algorithm presented in [7]. Also, building the auxiliary graph and testing if it is isomorphic to K_4 are linear time tasks. As was proved in [9], Step 4 may be performed in linear time as well, and Step 5 is also clearly linear. □

Corollary 10 *If G is $(claw, hourglass)$-free, then there exists a linear time algorithm to decide 3-colorability of G, and find a 3-coloring, if one exists.*

Proof: If $\Delta(G) \geq 5$, then G is not 3-colorable and this can be checked in linear time. Notice that the only disconnected neighborhoods that are allowed in a 3-colorable, claw-free graph are $2K_2$ and $K_1 \cup K_2$. Since G is hourglass-free, none of the neighborhoods is isomorphic to $2K_2$; otherwise, the graph induced by a vertex together with its neighborhood would be isomorphic to an hourglass. Hence, all neighborhoods in G are either connected or isomorphic to $K_1 \cup K_2$. Therefore, the graph is almost locally connected and by Theorem 9, the 3-colorability problem can be solved for it in linear time. □

The main result of this section raises the following natural question: is it possible to extend Theorem 9 to almost locally connected graphs of higher degree. Unfortunately, the answer is negative. This conclusion is beyond the scope of Section 3, which is devoted to positve results. Nevertheless, to make the subsection on almost locally connected graphs self-contained, we complete it with the proof of this negative result.

Theorem 11 *For almost locally connected graphs G with $\Delta(G) \geq 5$, the 3-COLORABILITY problem is NP-complete.*

Proof: Consider a graph H whose vertex set is $u, v_1, v_2, v_3, v_4, v_5, w_{12}, w_{23}, w_{34}, w_{45}$. Let $H[\{v_1, v_2, v_3, v_4, v_5\}]$ be a path, $w_{i,i+1}$ be adjacent to v_i, v_{i+1} for each $i = 1, 2, 3, 4$, and u be adjacent to every vertex in v_1, v_2, v_3, v_4, v_5. Notice that H is a 3-colorable, locally connected graph and vertices $w_{12}, w_{23}, w_{34}, w_{45}$ receive the same color in any 3-coloring of H.

Let G be any 4-regular graph on n vertices and G' a graph obtained by taking n copies of H - one corresponding to each vertex of G - and adding edges between copies of H in such a way that two copies of H are connected if and only if two vertices of G corresponding to them are, and there is exactly one edge connected to any of the vertices $w_{12}, w_{23}, w_{34}, w_{45}$.

It is easy to see that G' is an almost locally connected graph with $\Delta(G) = 5$. Moreover, G' is 3-colorable if and only if G is. Since for G, the 3-COLORABILITY problem is NP-complete, we conclude that it is NP-complete for G' too. □

3.2. *More general classes*

In this section we extend the result of the previous one to an infinitely increasing hierarchy of subclasses of claw-free graphs. The basis of this hierarchy is the class of $(claw, hourglass)$-free graphs. Now let us denote by H_k the graph obtained from a copy of an $hourglass$ H and a copy of P_k by identifying a vertex of degree 2 of H and a vertex of degree 1 of P_k. In particular, $H_1 = H$.

Theorem 12 *For any $k \geq 1$, there is a linear time algorithm to decide 3-colorability of a $(claw, H_k)$-free graph G, and to find a 3-coloring, if one exists.*

Proof: Without loss of generality we consider only connected K_4-free graphs of vertex degree at most 4 in the class under consideration. For those graphs that are $hourglass$-free, the problem is linear-time solvable by Corollary 10. Assume now that a $(claw, H_k)$-free graph G contains an induced hourglass H (which implies in particular that $k > 1$) and let v denote the center of H. There are only finitely many connected graphs of bounded vertex degree that do not have vertices of distance $k+1$ from v. Therefore, without loss of generality, we may suppose that G contains a vertex x_{k+1} of distance $k+1$ from v, and let $x_{k+1}, x_k, \ldots, x_1, v$ be a shortest path connecting x_{k+1} to v. In particular, x_1 is a neighbor of v. Since G is H_k-free, x_2 has to be adjacent to at least one more vertex, say u, in the neighborhood of v. If u is not adjacent to x_1, then u, x_1, x_2, x_3 induce a claw. Thus, u is adjacent to x_1, while x_2 has no neighbors in $N(v)$ other than x_1 and u.

Let us show that u has degree 3 in G. To the contrary, assume z is the fourth neighbor of u different from v, x_1, x_2. To avoid the induced claw $G[u, v, x_2, z]$, z has to be adjacent to x_2. This implies z is not adjacent to x_1 (else a $K_4 = G[x_1, u, x_2, z]$ arises), and z is adjacent to x_3 (else x_2, x_1, z, x_3 induce a claw). But now the path $x_k, \ldots, x_3, z, u$ together with v and its neighbors induce an H_k; a contradiction. By symmetry, x_1 also has degree 3 in G.

Notice that in any 3-coloring of G, vertices v and x_2 receive the same color. Therefore, by identifying these two vertices (more formally, by replacing them with a new vertex adjacent to every neighbor of v and x_2) and deleting u and x_1, we obtain a new graph G' which is 3-colorable if and only if G is. Moreover, it is not hard to see that the new graph is again $(claw, H_k)$-free. By applying this transformation repeatedly we reduce the initial graph to a graph G" which is either $(claw, hourglass)$-free or contains no vertices of distance $k + 1$ from the center of an hourglass. In both cases the problem is linear-time solvable, which completes the proof. □

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