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A full-Newton step infeasible interior-point algorithm for linear complementarity problems based on a kernel function

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Abstract

In this paper, we first present a brief infeasible interior-point method with full-Newton step for solving linear complementarity problem (LCP). The main iteration consists of a feasibility step and several centrality steps. First we present a full Newton step infeasible interior-point algorithm based on the classic logarithmical barrier function. After that a specific kernel function is introduced. Then the feasibility step is induced by this kernel function instead of the classic logarithmical barrier function. The results of complexity coincides with the best bound known for infeasible interior-point methods for LCP.

Key words: Interior-point methods, Linear complementarity problem, Full-Newton step, Complexity, Kernel functions.

1. Introduction

The linear complementarity problem (LCP) is to find a vector $(x, s) \in R^{2n}$ such that

$$(P) \qquad s = Mx + q, \ x, s \geq 0, \ \ xs = 0,$$

where $q \in R^n$, and $M \in R^{n\times n}$ is positive semidefinite. It is well known that many important problems in economics, control and game theory can be formulated as LCP. For a comprehensive study, the reader is referred to [1]. There are a variety of solution approaches for LCP which have been studied extensively. Among them, the interior-point methods (IPMs) gained much attention for LCP than other methods. For a comprehensive learning about IPMs, we refer to [2,18,20]. A close look at the IPM litterarure tells us that the first IPM for LCP was due to Kojima et al. [5] and their algorithm originated from the primal-dual IPMs for linear optimization (LO). Later on, Kojima et al. [6] set up a framework of IPMs for tracing the central path of a class of LCPs.

One may distinguish between feasible IPMs and infeasible IPMs (IIPMs). Feasible IPMs start with a strictly feasible interior point and maintain feasibility during the solution process. IIPMs start with an arbitrary positive point, and feasibilty is reached as optimality is approached. Potra [14] analyzed a generalization to LCPs of the Mizuno-Todd-Ye predictor corrector method [10] for infeasible starting points, and other literature about IIPMs for LCPs can be found in [14,19]. In Roos [17], a full Newton-step infeasible interior-point algorithm for LO was proposed and he also proved that the complexity of the algorithm coincides with the best known iteration bound for IIPMs. Some extensions on LO were carried out by Liu and Sun[7], Mansouri and Roos [9], on LCP by Mansouri et al. [8], and on semidefinite optimization by Kheirfam [3,4].

Recently, Peng et al. [11–13] proposed a new variant of IPMs based on self-regular kernel functions for LO and proved the best complexity for large-update methods with some specific self-regular kernel function.

Motivated by this series of work, we consider full Newton-step IIPM for LCP based on a specific kernel function. In our algorithm, we use a barrier function based on the simple kernel function

$$\phi(t) = \frac{1}{2}(t-1)^2, \tag{1}$$

instead of classical logarithmic barrier function to calculate the search directions. The complexity result shows that the full Newton-step IIPM for LCP based on this kernel function enjoys the best known iteration bound for LCP.

The paper is organized as follows: In the next Section, we present briefly primal-dual infeasible interior-point algorithm with a proximity of iterates (x, s) to the μ-center of the perturbed problem. In Section 3, we give some technically results. Section 4 is devoted to the analysis of the new feasibility step, which is the main

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part of the paper and then, iteration bound is derived. Finally, we end the paper in section 5.

2. The statement of the algorithm

Without loss of generality [6], we assume that LCP satisfies the interior point condition (IPC), i.e., there exists a vector $(x^0, s^0) \in R^{2n}$ such that;

$$s^0 = Mx^0 + q,\ x^0 > 0,\ s^0 > 0.$$

We assume that ρ_p and ρ_d are such that

$$\|x^*\|_\infty \leq \rho_p, \quad \max\{\|s^*\|_\infty, \rho_p\|Me\|_\infty, \|q\|_\infty\} \leq \rho_d,$$

for some optimal solution (x^*, s^*) of the problem (P) and start the algorithm with

$$x^0 = \rho_p e, \quad s^0 = \rho_d e, \quad \mu^0 = \rho_p\rho_d.$$

For any ν with $0 < \nu \leq 1$, we consider the perturbed problem

$$(P_\nu) \qquad s - Mx - q = \nu r^0, \quad (x,s) \geq 0,$$

where,

$$r^0 = s^0 - Mx^0 - q.$$

Note that if $\nu = 1$, then $(x,s) = (x^0, s^0)$ yields a strictly feasible solution of (P_ν). We conclude that if $\nu = 1$, then (P_ν) satisfies the IPC. More generally, one has the following result (see [8], Lemma 4.1).

Lemma 1. *If the original problem (P) is feasible then the perturbed problem (P_ν) satisfies the IPC for each $0 < \nu \leq 1$.*

Assuming that (P) is feasible, it follows from Lemma 1 that the problem (P_ν) satisfies the IPC, for each $0 < \nu \leq 1$. Then, its central path exists, meaning that the system

$$s - Mx - q = \nu r^0, \quad x \geq 0, s \geq 0, \tag{2}$$
$$xs = \mu e, \tag{3}$$

has a unique solution, for any $\mu > 0$. For $0 < \nu \leq 1$ and $\mu = \nu\mu^0$, we denote this unique solution in the sequel by $(x(\nu), s(\nu))$, where is the μ-center of (P_ν). In this notation, if we take $\nu = 1$, then $(x(1), s(1)) = (x^0, s^0)$. We measure the proximity of iterate (x,s) to the μ-center of the perturbed problem (P_ν) by the quantity

$$\delta(x,s;\mu) = \frac{1}{\sqrt{2}}\|v^{-1} - v\|, \tag{4}$$

where

$$v = \sqrt{\frac{xs}{\mu}}.$$

Initially, we have $\delta(x,s;\mu) = 0$. In the sequel, we assume that at start of each iteration, $\delta(x,s;\mu) \leq \tau$ with $\tau > 0$. This certainly holds at the start of the first iteration.

We now describe one main iteration of the algorithm given by Mansouri et al. [8]. The algorithm begins with an infeasible interior point (x,s) such that (x,s) is feasible for the perturbed problem (P_ν), $x^T s \leq (n+\delta^2)n$ and $\delta(x,s;\mu) \leq \tau$, where $\mu = \nu\mu^0$. Each main iteration consists of one so-called feasibility step, a μ-update, and a few centering steps. First we find a new point (x^f, s^f) which is feasible for the perturbed problem with $\nu^+ := (1-\theta)\nu$. Then μ is decreased to $\mu^+ := (1-\theta)\mu$. Generally, there is no guarantee that $\delta(x^f, s^f; \mu^+) \leq \tau$. So a limited number of centering steps are applied to produce a new point (x^+, s^+) such that $\delta(x^+, s^+; \mu^+) \leq \tau$, where $\mu^+ = \nu^+\mu^0$. This process is repeated until the algorithm terminates.

We now describe the search directions used in the feasibility and centering steps. For the feasibility step, the search direction $(\Delta^f x, \Delta^f s)$ defined by the system

$$M\triangle^f x - \triangle^f s = \theta\nu r^0, \tag{5}$$
$$s\triangle^f x + x\triangle^f s = \mu e - xs, \tag{6}$$

where, $\theta \in (0,1)$. If (x,s) is feasible for the perturbed problem (P_ν), then after the feasibility step the iterates satisfies the affine equation in (2), with $\nu = \nu^+$. Assuming that $\delta(x,s;\mu) \leq \tau$ holds before the step, and by taking θ small enough, it can be guaranteed that after the step the iterates

$$x^f = x + \triangle^f x, \quad s^f = s + \triangle^f s, \tag{7}$$

are positive and $\delta(x^f, s^f; \mu^+) \leq \frac{1}{\sqrt{2}}$.

In the centering step, starting at the iterate $(x,s) = (x^f, s^f)$ and targeting at the μ-center, the search direction $(\Delta x, \Delta s)$ is the usual primal-dual Newton direction, defined by

$$\begin{aligned} \triangle s &= M\triangle x, \\ s\triangle x + x\triangle s &= \mu e - xs. \end{aligned} \tag{8}$$

Denoting the iterate after a centering step by x^+ and s^+, we recall the following result from [8].

Lemma 2. *([8], Lemma 3.5, corollary 3.6) If $\delta := \delta(x,s;\mu) < 1$, then x^+ and s^+ are positive, $(x^+)^T s^+ \leq (n+\delta^2)\mu$. Moreover, if $\delta \leq \frac{1}{\sqrt{2}}$, then $\delta(x^+, s^+; \mu) \leq \delta^2$.*

Define

$$d_x^f := \frac{v\Delta^f x}{x}, \;\; d_x^f := \frac{v\Delta^f s}{s}, \tag{9}$$

where, v defined as (4). By using (9), we can rewrite (5)-(6) as follows

$$\begin{aligned} MS^{-1}Xd_x^f - d_s^f &= \theta\nu v s^{-1} r^0, \\ d_x^f + d_s^f &= v^{-1} - v, \end{aligned} \tag{10}$$

where $X = \text{diag}(x),\ S = \text{diag}(s)$.
Note that the right-hand-side in the second equation of system (10), $v^{-1} - v$, equals the negative gradient of the classical logarithmic barrier function

$$\Psi(v) := \sum_{i=1}^{n} \psi(v_i), \quad v_i = \sqrt{\frac{x_i s_i}{\mu}},$$

whose kernel function is

$$\psi(t) = \frac{1}{2}(t^2 - 1) - \log t.$$

The main contribution of this paper is a modification of the feasibility step. We present a slightly algorithm, obtained by changing the definition of the feasibility step via replacing the second equation of (10) by $d_x^f + d_s^f = -\nabla\phi(v)$, where the kernel function of $\phi(v)$ defined as (1). Therefore, the system of the new feasibility step becomes

$$\begin{aligned} MS^{-1}Xd_x^f - d_s^f &= \theta\nu v s^{-1} r^0, \\ d_x^f + d_s^f \; &= -\nabla\phi(v). \end{aligned}$$

Since $\phi^{'}(t) = t - 1$, the second equation in the system can be written as

$$d_x^f + d_s^f = e - v. \tag{11}$$

We define

$$\sigma(x, s; \mu) := \sigma(v) := \frac{1}{\sqrt{2}}\|e - v\|. \tag{12}$$

It is obvious that $\sigma(v) = 0$ if and only if $v = e$, thus $\sigma(v)$ is also a suitable proximity. We now give a more formal description of Algorithm 1 below.

Algorithm1 : $Primal - Dual\ InfeasibleIPM$

Input :
  Accuracy parameter$\epsilon > 0$;
  barrier update parameter $\theta,\ 0 < \theta < 1$;
  feasible $(\mathrm{x}^0, \mathrm{s}^0)$ with $(\mathrm{x}^0)^{\mathrm{T}}\mathrm{s}^0 = \mathrm{n}\mu$,
  $\delta(\mathrm{x}^0, \mathrm{s}^0; \mu) < \tau = \frac{1}{2}$.
begin
    $x := x^0;\ s := s^0;\ \mu := \mu^0;$
  while $\max(n\mu, \|r\|) > \epsilon$ **do**
   begin
    feasibility step :$(x, s) := (x, s) + (\triangle^f x, \triangle^f s)$;
    $\mu -$ update :$\mu := (1 - \theta)\mu$;
    centering steps :
    while $\delta(x, s; \mu) \geq \tau$ **do**
     begin
      $(x, s) := (x, s) + (\Delta x, \Delta s)$;
     end
   end
end.

3. Technical results

We now give some lemmas which are used in the analysis later.

Lemma 3. *(Lemma 3 in [21]) Suppose that $\psi(x) = \psi_1(x)+\psi_2(x)$, both $\psi_1(x)$ and $\psi_2(x)$ are strictly monotone increasing in a given interval. The roots of $\psi_1(x) = 0$ and $\psi_2(x) = 0$ are x_1 and x_2 respectively. Then the root x^* of $\psi(x) = 0$ satisfies that*

$$x^* \geq \min\{x_1, x_2\}.$$

Lemma 4. *([9], Lemma A.1) For $i = 1, 2, \ldots, n$, let $f_i : \mathbb{R}_+ \to \mathbb{R}$ denote a convex function. Then, for any nonzero vector $z \in \mathbb{R}_+^n$, the following inequality*

$$\sum_{i=1}^{n} f_i(z_i) \leq \frac{1}{e^T z} \sum_{j=1}^{n} z_j \left(f_j(e^T z) + \sum_{i \neq j} f_i(0) \right)$$

holds.

Lemma 5. *(Lemma 5.6 in [8]) The iterates (x^f, s^f) are certainly strictly feasible if*

$$\frac{1}{q(\delta)} \leq v_i \leq q(\delta), \quad i = 1, 2, \ldots, n \tag{13}$$

where

$$q(\delta) = \frac{\sqrt{2}}{2}\delta + \sqrt{\frac{1}{2}\delta^2 + 1}.$$

Lemma 6. *(Lemma 5.8 in [8]) Let (x,s) be feasible for the perturbed problem (P_ν) and $(x^0,s^0)=(\rho_p e,\rho_d e)$. Then*

$$\|x\|_1 \le \big(2+q(\delta)^2\big)n\rho_p,$$

where, $q(\delta)$ is defined in Lemma 5.

The following lemma shows the effect on the proximity measure if v is replaced by $\tilde{v} := \sqrt{\frac{v}{1-\theta}}$.

Lemma 7. *Let (x,s) be a primal-dual Newton step and $\mu>0$ such that $x^Ts \le (n+\delta^2)\mu$. Moreover let $\delta(v):=\delta(x,s;\mu)$ and $\tilde{v} := \dfrac{v^{\frac{1}{2}}}{\sqrt{1-\theta}}$. Then*

$$\delta(\tilde{v})^2 \le 2(1-\theta)\delta^2 + \frac{\theta(2-\theta)}{1-\theta}\sqrt{n(n+\delta^2)}$$

.

Proof. By using $\|v\|^2 \le n+\delta^2$ and Holder inequality, one has

$$\|v^{\frac{1}{2}}\|^2 = \sum_{i=1}^n v_i \le (n\sum_{i=1}^n v_i^2)^{\frac{1}{2}} \le \sqrt{n(n+\delta^2)}.$$

By following definition $\delta(v)$ and the clear inequality, $|t^{-\frac{1}{2}} - t^{\frac{1}{2}}| \le |t^{-1}-t|$, for $t>0$ we have

$$\begin{aligned}
2\delta(\tilde{v})^2 &= \|\sqrt{1-\theta}v^{-\frac{1}{2}} - \frac{v^{\frac{1}{2}}}{\sqrt{1-\theta}}\|^2 \\
&= \Big\|\sqrt{1-\theta}(v^{-\frac{1}{2}} - v^{\frac{1}{2}}) - \frac{\theta v^{\frac{1}{2}}}{\sqrt{1-\theta}}\Big\|^2 \\
&= (1-\theta)\|v^{-\frac{1}{2}} - v^{\frac{1}{2}}\|^2 + \frac{\theta^2\|v^{\frac{1}{2}}\|^2}{1-\theta} \\
&\quad - 2\theta\big(v^{-\frac{1}{2}} - v^{\frac{1}{2}}\big)^T v^{\frac{1}{2}} \\
&\le 2(1-\theta)\delta(v)^2 + \frac{\theta^2\sqrt{n(n+\delta^2)}}{1-\theta} \\
&\quad + 2\theta\Big(\|v^{\frac{1}{2}}\|^2 - n\Big) \\
&\le 2(1-\theta)\delta(v)^2 + \frac{\theta^2\sqrt{n(n+\delta^2)}}{1-\theta} \\
&\quad + 2\theta\Big(\sqrt{n(n+\delta^2)} - n\Big) \\
&\le 2(1-\theta)\delta^2 + \frac{\theta(2-\theta)}{1-\theta}\sqrt{n(n+\delta^2)}.
\end{aligned}$$

□

4. Analysis

Let (x,s) denote the iterate at start of an iteration with $x^Ts \le \mu(n+\delta^2)$ and $\delta(x,s;\mu)\le\tau$. Recall that at the start of the first iteration this certainly holds, because $\delta(x,s;\mu)=0$.

4.1. *Effect of the feasibility step*

According to Lemma 2, we need to show that $\delta(x^f,s^f;\mu^+) \le \frac{1}{\sqrt{2}}$ after the feasibility step, that is, that the new iterates are within the region, where the Newton process targeting at the μ^+-center of (P_{ν^+}) is quadratically convergent. Using (9), (11) and $xs=\mu v^2$, we obtain

$$\begin{aligned}
x^f s^f &= xs + (s\Delta^f x + x\Delta^f s) + \Delta^f x\Delta^f s \\
&= \mu v^2 + \mu v(d_x^f + d_s^f) + \mu d_x^f d_s^f \\
&= \mu(v + d_x^f d_s^f). \qquad (14)
\end{aligned}$$

The next lemma gives conditions for strict feasibility of the full Newton-step.

Lemma 8. *The iterates (x^f,s^f) are strictly feasible if and only if $v+d_x^f d_s^f>0$.*

Proof. Note that if x^f and s^f are positive then (14) makes clear that $v+d_x^f d_s^f>0$. For the proof of the converse, introduce a step length α with $0\le\alpha\le1$ and define

$$x^\alpha = x+\alpha\Delta^f x, \quad s^\alpha = s+\alpha\Delta^f s.$$

We thus have $x^0=x, x^1=x^f, s^0=s$ and $s^1=s^f$. Note that $x^0s^0=xs>0$. We may write

$$\begin{aligned}
x^\alpha s^\alpha &= (x+\alpha\Delta^f x)(s+\alpha\Delta^f s) \\
&= xs + \alpha(s\Delta^f x + x\Delta^f s) + \alpha^2\Delta^f x\Delta^f s. \qquad (15)
\end{aligned}$$

From (9), we deduce $\Delta^f x\Delta^f s = \mu d_x^f d_s^f$. Using this and $x\Delta^f s + s\Delta^f x = \mu v(d_x^f + d_s^f) = \mu v(e-v)$ and $\mu v^2 = xs$, we obtain

$$x^\alpha s^\alpha = \mu\big((1-\alpha)v^2 + \alpha(v+\alpha d_x^f d_s^f)\big). \qquad (16)$$

If $v+d_x^f d_s^f>0$, then $d_x^f d_s^f > -v$. Substituting into (16), we get

$$\begin{aligned}
&x^\alpha s^\alpha > \\
&\quad \mu\big((1-\alpha)v^2 + \alpha(v-\alpha v)\big) = \mu(1-\alpha)(v^2+\alpha v).
\end{aligned}$$

Since $\mu(1-\alpha)(v^2+\alpha v)\ge0$, it follows that $x^\alpha s^\alpha>0$ for $0\le\alpha\le1$. Hence, none of the entries of x^α and s^α vanish for $0\le\alpha\le1$. Since $x^0=x>0$ and $s^0=s>0$, and x^α and s^α depend linearly on α, this implies that $x^\alpha>0$ and $s^\alpha>0$ for $0\le\alpha\le1$. Hence x^1 and s^1 must be positive, and the proof is complete. □

In the sequel, we denote

$$w_i := w_i(v) := \frac{1}{2}\sqrt{|d_{x_i}^f|^2 + |d_{s_i}^f|^2},$$

and

$$w := w(v) := ||(w_1, w_2, \ldots, w_n)||.$$

These imply, by Cauchy-Schwartz inequality,

$$(d_x^f)^T d_s^f \leq ||d_x^f||\ ||d_s^f|| \leq \frac{1}{2}(||d_x^f||^2 + ||d_s^f||^2) \leq 2w^2,$$

$$|d_{x_i}^f d_{s_i}^f| = |d_{x_i}^f||d_{s_i}^f| \leq \frac{1}{2}\Big(|d_{x_i}^f|^2 + |d_{s_i}^f|^2\Big) \leq 2w_i^2 \leq 2w^2, \quad i = 1, 2, \ldots, n.$$

Theorem 9. *If* $v + d_x^f d_s^f > 0$ *and* $1 - 2q(\delta)w^2 > 0$*, then*

$$2\delta(v^f)^2 \leq 2(1-\theta)\delta^2 + \frac{2w^2}{1-\theta} + \frac{2(1-\theta)w^2 q(\delta)^2}{1 - 2w^2 q(\delta)} + \frac{\theta(2-\theta)}{1-\theta}\sqrt{n(n+\delta^2)}.$$

Proof. Using (14), after division of both sides by $\mu^+ = (1-\theta)\mu$ we obtain

$$(v^f)^2 = \frac{\mu\big(v + d_x^f d_s^f\big)}{\mu^+} = \frac{v + d_x^f d_s^f}{1-\theta}.$$

Hence

$$2\delta(v^f)^2 = \sum_{i=1}^n \Big((v_i^f)^2 + (v_i^f)^{-2} - 2\Big) = \sum_{i=1}^n \Big(\frac{v_i + d_{x_i}^f d_{s_i}^f}{1-\theta} + \frac{1-\theta}{v_i + d_{x_i}^f d_{s_i}^f} - 2\Big) \leq \sum_{i=1}^n \Big(\frac{v_i + 2w_i^2}{1-\theta} + \frac{1-\theta}{v_i - 2w_i^2} - 2\Big).$$

Define

$$f_i(2w_i^2) = \frac{v_i + 2w_i^2}{1-\theta} + \frac{1-\theta}{v_i - 2w_i^2} - 2, \quad i = 1, 2, \ldots, n.$$

Using Lemma 5 and the hypothesis $1 - 2q(\delta)w^2 > 0$, we obtain

$$v_i - 2w_i^2 > 0.$$

This implies that $f_i(2w_i^2)$ is convex in $2w_i^2$. Therefore, by Lemma 4,

$$2\delta(v^f)^2 \leq \sum_{j=1}^n f_j(2w_j^2) \leq \frac{1}{2w^2}\sum_{j=1}^n 2w_j^2\Big(f_j(2w^2) + \sum_{i\neq j} f_i(0)\Big)$$
$$= \frac{1}{2w^2}\sum_{j=1}^n 2w_j^2\Big[\Big(\frac{v_j + 2w^2}{1-\theta} + \frac{1-\theta}{v_j - 2w^2} - 2\Big) + \sum_{i\neq j}\Big(\frac{v_i}{1-\theta} + \frac{1-\theta}{v_i} - 2\Big)\Big].$$

Using Lemma 7, we obtain

$$\sum_{i\neq j}\Big(\frac{v_i}{1-\theta} + \frac{1-\theta}{v_i} - 2\Big) = \sum_{i=1}^n \Big(\frac{v_i}{1-\theta} + \frac{1-\theta}{v_i} - 2\Big) - \Big(\frac{v_j}{1-\theta} + \frac{1-\theta}{v_j} - 2\Big)$$
$$\leq 2(1-\theta)\delta^2 + \frac{\theta(2-\theta)}{1-\theta}\sqrt{n(n+\delta^2)} - \Big(\frac{v_j}{1-\theta} + \frac{1-\theta}{v_j} - 2\Big).$$

Therefore,

$$2\delta(v^f)^2 \leq \frac{1}{2w^2}\sum_{j=1}^n 2w_j^2\Big[\Big(\frac{v_j + 2w^2}{1-\theta} + \frac{1-\theta}{v_j - 2w^2} - 2\Big) - \Big(\frac{v_j}{1-\theta} + \frac{1-\theta}{v_j} - 2\Big)\Big]$$
$$+ \frac{1}{2w^2}\sum_{j=1}^n 2w_j^2\Big(2(1-\theta)\delta^2 + \frac{\theta(2-\theta)}{1-\theta}\sqrt{n(n+\delta^2)}\Big)$$
$$= \frac{1}{2w^2}\sum_{j=1}^n 2w_j^2\Big[\frac{2w^2}{1-\theta} + \frac{1-\theta}{v_j - 2w^2} - \frac{1-\theta}{v_j}\Big] + 2(1-\theta)\delta^2 + \frac{\theta(2-\theta)}{1-\theta}\sqrt{n(n+\delta^2)}$$
$$= \frac{2w^2}{1-\theta} + (1-\theta)\sum_{j=1}^n 2w_j^2 \frac{1}{v_j\big(v_j - 2w^2\big)} + 2(1-\theta)\delta^2 + \frac{\theta(2-\theta)}{1-\theta}\sqrt{n(n+\delta^2)}$$
$$\leq \frac{2w^2}{1-\theta} + 2(1-\theta)w^2\frac{q(\delta)^2}{1-2w^2q(\delta)} + 2(1-\theta)\delta^2 + \frac{\theta(2-\theta)}{1-\theta}\sqrt{n(n+\delta^2)}.$$

This completes the proof. □

We conclude this section by presenting a value that we do not allow w to exceed. Needing $\delta(v^f) \leq \frac{1}{\sqrt{2}}$, it follows from Lemma 9 that it is sufficient to have

$$2(1-\theta)\delta^2 + \frac{2w^2}{1-\theta} + \frac{2(1-\theta)w^2q(\delta)^2}{1-2w^2q(\delta)} + \frac{\theta(2-\theta)}{1-\theta}\sqrt{n(n+\delta^2)} \leq 1.$$

Now,

$$\psi(t) = \frac{2t}{1-\theta} + \frac{2(1-\theta)q(\delta)^2 t}{1-2q(\delta)t} + 2(1-\theta)\delta^2 + \frac{\theta(2-\theta)}{1-\theta}\sqrt{n(n+\delta^2)} - 1,$$

and

$$\psi_1(t) = \frac{2t}{1-\theta} + 2(1-\theta)\delta^2 + \frac{\theta(2-\theta)}{1-\theta}\sqrt{n(n+\delta^2)} - \frac{1}{2},$$

$$\psi_2(t) = 2(1-\theta)t\frac{q(\delta)^2}{1-2tq(\delta)} - \frac{1}{2}.$$

Note that $\psi(t) = \psi_1(t) + \psi_2(t)$, and that both $\psi_1(t)$ and $\psi_2(t)$ are monotonically increasing in t. By Lemma 3, the root t^* of $\psi(t) = 0$ satisfies $t^* \geq \min\{t_1^*, t_2^*\}$, where t_1^* and t_2^* are the roots of $\psi_1(t) = 0$ and $\psi_2(t) = 0$, respectively.

Since $\psi_1(t_1^*) = 0$,

$$t_1^* = \left(\frac{1}{2} - 2(1-\theta)\delta^2 - \frac{\theta(2-\theta)}{1-\theta}\sqrt{n(n+\delta^2)}\right)\frac{1-\theta}{2},$$

and from $\psi_2(t_2^*) = 0$,

$$t_2^* = \frac{1}{4(1-\theta)q(\delta)^2 + 2q(\delta)}.$$

At this stage, we choose

$$\tau = \frac{1}{8}, \quad \theta = \frac{\alpha}{8n}, \ \alpha \leq 1.$$

Then for $n \geq 2$ and $\delta \leq \tau$, it can easily be verified that

$$t_1^* \geq \frac{5}{32} - \frac{\alpha}{8}\sqrt{1 + \frac{\delta^2}{n}} + \frac{\alpha^2}{132}\sqrt{\frac{n+\delta^2}{n^3}} \geq \frac{25}{256}, \quad (17)$$

$$t_2^* \geq \frac{4}{31}. \quad (18)$$

Using (17) and the assumption that $1 - 2q(\delta)w^2 > 0$, it is easily verified that if

$$w^2 \leq \min\{\frac{25}{256}, \frac{4}{31}, \frac{4}{9}\} = \frac{25}{256}, \quad (19)$$

then

$$\delta(v^f) \leq \frac{1}{\sqrt{2}}.$$

4.2. *Upper bound for* $\|d_x^f\|^2 + \|d_s^f\|^2$

In this section, we obtain an upper bound for $\|d_x^f\|^2 + \|d_s^f\|^2$, which enables us to find a default value for θ. We consider the following system:

$$\begin{aligned} MS^{-1}Xd_x^f - d_s^f &= \theta\nu vs^{-1}r^0 \\ d_x^f + d_s^f &= e - v, \end{aligned} \quad (20)$$

where $X = \operatorname{diag}(x)$ and $S = \operatorname{diag}(s)$. By eliminating d_s^f from (20), we have

$$d_x^f = \left(I + MS^{-1}X\right)^{-1}\left(e - v + \theta\nu vs^{-1}r^0\right). \quad (21)$$

Since $I + MS^{-1}X$ is positive definite, it follows

$$\|d_x^f\| \leq \|e - v + \theta\nu vs^{-1}r^0\|. \quad (22)$$

Hence, by using (20), Cauchy-Schwartz inequality and positive semidefiniteness of $MS^{-1}X$ we have

$$\begin{aligned} \|d_x^f\|^2 + \|d_s^f\|^2 &= \|d_x^f + d_s^f\|^2 - 2(d_x^f)^T d_s^f \\ &= \|e - v\|^2 \\ &- 2(d_x^f)^T(MS^{-1}Xd_x^f - \theta\nu vs^{-1}r^0) \\ &= \|e - v\|^2 \\ &-2\,(d_x^f)^T MS^{-1}Xd_x^f + 2\theta\nu(d_x^f)^T vs^{-1}r^0 \\ &\leq \|e - v\|^2 + 2\|d_x^f\|\|\theta\nu vs^{-1}r^0\|. \end{aligned}$$

Using (22), we get

$$\begin{aligned} &\|d_x^f\|^2 + \|d_s^f\|^2 \\ &\leq \|e - v\|^2 + 2\|\left(e - v + \theta\nu vs^{-1}r^0\right)\|\|\theta\nu vs^{-1}r^0\| \\ &\leq \|e - v\|^2 + 2\left(\|e - v\| + \|\theta\nu vs^{-1}r^0\|\right)\|\theta\nu vs^{-1}r^0\| \\ &\leq 2\delta^2 + 2\left(\sqrt{2}\delta + \frac{3\theta}{\rho_p v_{\min}}\|x\|_1\right)\frac{3\theta}{\rho_p v_{\min}}\|x\|_1. \end{aligned}$$

By using Lemmas 5 and 6, we obtain

$$\begin{aligned} &\|d_x^f\|^2 + \|d_s^f\|^2 \leq \\ &2\delta^2 + 6n\theta\left(\sqrt{2}\delta + 3n\theta q(\delta)\left(2 + q(\delta)^2\right)\right)q(\delta)\left(2 + q(\delta)^2\right). \end{aligned}$$

4.3. *Value for* θ

At this stage, we choose

$$\tau = \frac{1}{8}. \quad (23)$$

Since $\delta \leq \tau = \frac{1}{8}$ and $q(\delta)$ is monotonically increasing in δ,

$$\begin{aligned} &\|d_x^f\|^2 + \|d_s^f\|^2 \leq 2\delta^2 \\ &+6n\theta\left(\sqrt{2}\delta + 3n\theta q(\delta)\left(2 + q(\delta)^2\right)\right)q(\delta)\left(2 + q(\delta)^2\right) \\ &\leq 2(\frac{1}{8})^2 \\ &+6n\theta\left(\frac{\sqrt{2}}{8} + 3n\theta q(\frac{1}{8})\left(2 + q(\frac{1}{8})^2\right)\right)q(\frac{1}{8})\left(2 + q(\frac{1}{8})^2\right) \\ &\leq \frac{1}{32} + \left(0.18 + 10.46n\theta\right)20.92n\theta. \end{aligned}$$

Using $\theta = \dfrac{\alpha}{8n}$, the above inequality reads

$$\|d_x^f\|^2 + \|d_s^f\|^2 \leq \frac{1}{32} + (0.18 + 1.31\alpha)2.62\alpha. \quad (24)$$

From (19), we know that $w^2 \leq \frac{25}{256}$ is needed to guarantee that $\delta(v^f) \leq \frac{1}{\sqrt{2}}$. By (24), this holds if $(0.18 + 1.31\alpha)2.62\alpha \leq 0.36$. This means if we take

$$\alpha = 0.25, \quad (25)$$

then it is guaranteed that $\delta(v^f) \leq \frac{1}{\sqrt{2}}$.

4.4. *Complexity analysis*

We have seen that if the iterates satisfy $\delta(x, s; \mu) \leq \tau$ at the start of an iteration, with τ as defined in (23), then after the feasibility step, with $\theta = \frac{\alpha}{8n}$ and α as defined in (25), the iterates satisfy $\delta(x^f, s^f; \mu^+) \leq \frac{1}{\sqrt{2}}$.

After the feasibility step, we preform some centering steps in order to make the iterate (x^+, s^+) that satisfies $\delta(x^+, s^+; \mu^+) \leq \tau$, where τ is smaller than $\frac{1}{\sqrt{2}}$. This process is repeated until the maximum of the norm of the residual and $(x^+)^T s^+$ is less than ϵ.

The next theorem gives an upper bound for the total number of iterations, which this bound coincides with the currently best bound of infeasible IPMs for LCP except a factor [15,16].

Theorem 10. *If (P) has optimal solution (x^*, s^*) such that $\|x^*\|_\infty \leq \rho_p$ and $\|s^*\|_\infty \leq \rho_d$, then after at most*

$$128n \log \frac{\max\{(x^0)^T s^0, \|r^0\|\}}{\epsilon},$$

iterations the algorithm finds an ϵ-optimal solution of (P).

Proof. Let k denote the number of centering steps. By Lemma 2, after k centering steps, the iterate (x^+, s^+) is still feasible for (P_{ν^+}) and satisfies

$$\delta(x^+, s^+; \mu^+) \leq \left(\frac{1}{\sqrt{2}}\right)^{2^k}.$$

From this inequality, one can easily deduces that $\delta(x^+, s^+; \mu^+) \leq \tau$ will hold after at most

$$\log_2\left(\log_2 \frac{1}{\tau^2}\right) = \log_2(\log_2 64) = 3, \quad (26)$$

centering steps. So each iteration consists of at most 4 so-called 'inner' iterations, in each of which we need to compute a new search direction. In each main iteration both the value of $x^T s$ and the norm of residual are reduced by the factor $1 - \theta$. Hence, the total number of main iterations is bounded above by

$$\frac{1}{\theta} \log \frac{\max\{(x^0)^T s^0, \|r^0\|\}}{\epsilon}.$$

By multiplying the number of inner iterations by the number of main iterations, the total number of iterations is bounded above by

$$\frac{4}{\theta} \log \frac{\max\{(x^0)^T s^0, \|r^0\|\}}{\epsilon},$$

where $\theta = \frac{1}{32n}$, by (25), and hence we obtain the upper bound

$$128n \log \frac{\max\{(x^0)^T s^0, \|r^0\|\}}{\epsilon}.$$

□

5. Conclusion

In this paper we extended the full Newton-step infeasible interior-point algorithm to LCP. After that we used a new kernel function to induce the feasibility step and we analyzed the algorithm based on this kernel function. The same results of complexity are obtained.

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