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Minimum Cost Flow Problem on Dynamic Multi Generative Networks

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Abstract

This paper consists in constructing and modeling Dynamic Multi Generative Network Flows in which the flow commodities is dynamically generated at source nodes and dynamically consumed at sink nodes. It is assumed that the source nodes produce the flow commodities according to k time generative functions and the sink nodes absorb the flow commodities according to k time consumption functions. The minimum cost dynamic flow problem in such networks that extend the classical optimal flow problems on static networks, for a pre-specified time horizon T is defined and mathematically formulated. Moreover, it is showed that the dynamic problem on these networks can be formulated as a linear program whose special structure permits efficient computations of its solution and can be solved by one minimum cost static flow computation on an auxiliary time-commodity expanded network. By using flow decomposition theorem, we elaborate a different model of the problem to reduce its complexity. We consider the problem in the general case when the cost and capacity functions depend on time and commodity.

Key words: Network/graphs, LP problems, Decomposition methods.

1. Introduction

Dynamic flow problems are among the most important and challenging problems in network optimization. Dynamic flows are widely used in modeling of control processes from different technical, economic and informational systems. Road or air traffic control, production systems, evacuation planning, scheduling planning, economic planning, telecommunication, transportation, communication, and management problems can be formulated and solved as single-commodity or multi-commodity flow problems ([1, 3, 7, 8, 10]). Ford and Fulkerson introduced ([5, 18]) flows over time to add a time dimension to the traditional network flow models. Given a network with capacities and transit times on arcs, they studied the problem of sending a maximum amount of flow from a source node s to a sink node d within a pre-specified time horizon T. They showed that this problem can be solved by one minimum cost static flow computation, where transit times on arcs are interpreted as cost coefficients. Since then, optimal dynamic flow problems have attracted many researchers for several different reasons such as large practical applicability of these problems and their relationship with different combinatorial problems. Subsequently linear models of dynamic flow problems have been studied by M. Skutella, L. Fleischer, G. Glockner, B. Hoppe, J. Nemhauser, E. Subrahmanian and others in [9, 11, 12, 14, 16, 17, 20, 22, 23]. In all of the existing results on dynamic network flows that we have mentioned above, the time dimension has been employed for flow transit times, but in many optimization problems originating from real-life systems, the time factor over generations or consumptions of flow is a key variable to the problem formulation [24]. In the classical network flow theory, however, this factor is not sufficiently reflected. For that reason, in this paper, multicommodity version of Dynamic Generative Network Flows (DGNF), which provide an adequate framework for modeling such network-structured problems, are analyzed. Dynamic multi (multi-commodity) generative network flows are considered with time-commodity varying capacities of arcs and also for the minimum cost flow problem it is assumed that cost function depends on time and commodity.

This paper consists in modeling, investigation and solving the problem of minimum cost dynamic flow (MCDF) on a dynamic multi generative network flow $N = (V, \mathrm{A}, k, T)$ with different forms of restrictions by parameters of network and time. We will assign to each node $i \in V$, k time functions. We call this kind of

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network flows multi generative dynamic network flows (DMGNF) where travel and transmission are instantaneous. In these networks the supplies and demands are not given as fixed values; instead, they are continuously generated. Therefore, in such kind of network flows, the amount of flow of any product on each arc changes at every moment of time during a pre-specified time horizon T. The above noted aspects of these networks distinguish them from the multicommodity static and transmission time dependent network flows [24].

In this paper we investigate multi generative version of the MCDF problem on a generative network. The problem of minimum cost dynamic multi generative network flow consists of shipping several different products generated by several different generative functions (in source nodes) to sink nodes through a given network satisfying certain objectives in such a way that the total flow going through arcs does not exceed their capacities. No commodity ever transforms into another commodity at no moment of time, so that each one has its own flow conservation constraints during the time horizon T. We consider the minimum cost flow problem on multi generative networks with time-commodity varying capacities and costs on arcs that depend on time and commodity entering them.

This paper proposes two methods for studying and solving the MCDF problem. The basic method used for investigation, rely on a time-commodity expanded network. As it will be showed, the MCDF problem on DMGNF can be formulated as a linear programming model whose special structure permits efficient computation of its solution. In the next step we elaborate a different model of the problem in order to decrease its complexity.

In section 2 we explain the needed definitions and preliminary items. The MCDF problem in a multi generative network is developed in two cases, continuous and discrete time, in section 3. A method to solve the problem is explained in section 4 and finally, an efficient model in order to decrease the problem complexity is presented in section 5.

2. Definitions and Preliminaries

In this section we provide basic definitions and preliminary items on multi generative dynamic network flows. Assume that $N = (V, \mathrm{A}, k, T)$ is a DMGNF with node set V, arc set A and integer time horizon T. For each node $i \in V$ is either k different generative functions $p_{i1}(t), p_{i2}(t), ..., p_{ik}(t)$ or k different consumption functions $r_{i1}(t), r_{i2}(t), ..., r_{ik}(t)$, where $p_{iq}(t)$ is a time function that produces $p_{iq}(t)$ units of flow of commodity q at time $t \in [0, T]$, similarly $r_{iq}(t)$ is a function of time and shows the amount of flow of commodity q that node i consumes at time $t \in [0, T]$. Here, k is the number of types of generative functions which defines set $\mathrm{K} = \{1, 2, ..., k\}$ consist in types of products generated by generative functions (on the other side, types of products consumed by consumption functions) that must be routed through the network. Time horizon T is the time until which the flow can travel in the network. Also we assign to each arc $(i, j) \in \mathrm{A}$ non-negative capacities $u_{ijq}(t)$ and $u_{ij}(T)$, which represent the maximum amount of flow of product q at moment t that can be carried on (i, j) and the maximum amount of total flow (all of the products) that can be carried on (i, j) during the time horizon T, respectively .

We may assign to node$i \in V$,k different generative functions$p_{iq}(t)$, in this case we will refer to this node as a multi generative node. In the other case we may assign to node$i \in V$, k different consumption functions $r_{iq}(t)$, in this case we regard i as a multi consumer node. In the other cases we consider i as a transshipment node, in this case $p_{iq}(t) = r_{iq}(t) = 0$ for $q = 1, 2, ..., k$ and for each $t \in [0, T]$.

In these networks, we have several important properties that are the key factors to model the problem, as follows:

1. The generative functions $p_{iq}(t)$ for every $i \in V$and$t \in [0, T]$ transmit the same commodity q into the network.
2. The consumption functions $r_{iq}(t)$ for every $i \in V$and $t \in [0, T]$ absorb the same commodity$q$.
3. Each node i must be contained k time functions $p_{i1}(t), p_{i2}(t), ..., p_{ik}(t)$ or k time functions $r_{i1}(t), r_{i2}(t), ..., r_{ik}(t)$ but no of both types. Hence,

3.1. If a multi generative node contains $r < k$ generative functions for r commodities, we can assign $k - r$ generative functions for other commodities with zero value.

3.2. If a multi consumer node contains $r < k$ consumption functions for r commodities, we can assign $k - r$ consumption functions for other commodities with zero value.

The amount of generated flow in multi generative nodes and the amount of absorbed flow in multi consumer nodes are functions of time, and so, the flow value on all arcs connecting these nodes will change at every moment of time in$[0, T]$. We define the power of multi generative node i respect to product q at time horizon T as:

$$P_{iq}(T) = \int_0^T p_{iq}(t)dt,$$

and the total power of multi generative node i at time horizon T as:

$$P_i(T) = \sum_{q\in \mathrm{K}} P_{iq}(T).$$

Similarly, we define the power of multi consumer node i respect to product q at time horizon T as:

$$R_{iq}(T) = \int_0^T r_{iq}(t)dt,$$

and the total power of multi consumer node i respect to product q at time horizon T as:

$$R_i(T) = \sum_{q\in \mathrm{K}} R_{iq}(T).$$

For convenience in modeling, we define for every node i potential energy function $v_{iq}(t)$ as: $v_{iq}(t) = p_{iq}(t)$ for every multi generative node i respect to commodity q, and $v_{iq}(t) = -r_{iq}(t)$ for every multi consumer node i respect to commodity q. Therefore, $V_i(T) = P_i(T)$ is the total potential energy of multi generative node i at time horizon T and $V_i(T) = -R_i(T)$ is the total potential energy of multi consumer node i at time horizon T. In a dynamic multi generative network, $x(t) : \mathrm{A} \rightarrow R^+$ is a feasible dynamic flow, if it satisfies the following constraints:

$$\sum_j \int_0^\theta x_{ijq}(t)dt - \sum_j \int_0^\theta x_{jiq}(t)dt = \int_0^\theta v_{iq}(t)dt \forall i \in V, \forall q \in \mathrm{K}, \forall \theta \in [0,T] \quad (1)$$

$$0 \leqslant x_{ijq}(t) \leqslant u_{ijq}(t) \forall (i,j) \in \mathrm{A}, \forall q \in \mathrm{K}, \forall t \in [0,T] \quad (2)$$

$$0 \leqslant \sum_{q\in \mathrm{K}} \int_0^T x_{ijq}(t)dt \leqslant u_{ij}(T) \forall (i,j) \in \mathrm{A} \quad (3)$$

$$x_{ijq}(t) = 0 \forall (i,j) \in \mathrm{A}, \forall q \in \mathrm{K}, \forall t > T \quad (4)$$

Where $x_{ijq}(t)$ is the amount of flow of commodity q passing arc (i,j) at moment t. Conditions (1) are the flow conservation constraints that we require flow to be balanced at every time moment for every node with respect to each commodity (i.e., it is required that flow to be balanced during the time horizon for every node with respect to each commodity). Conditions (2) and (3) are the conditions of flow feasibility for every moment t and horizon T with respect to each commodity. In other words, condition (2) represents the maximum amount of flow of any commodity that can be carried on (i,j) at every moment t and Condition (3) represents the total flow of all commodities that can be carried on (i,j) during the time horizon T. Condition (4) guarantees flow can just travel in the network until the end of pre-specified time horizon.

3. The Minimum Cost Dynamic Flow Problem on Dynamic Multi Generative Network Flow

We can formulate the dynamic flow problem in two ways depending on whether we use a discrete or continuous representation of time. The discrete-time dynamic flow problem is a discrete-time expansion of a static network flow problem. In this case we distribute the flow over a set of predetermined time steps $t = 0, 1, ..., T-1$. In a continuous-time dynamic flow problem we look for the flow which distributed continuously over time within the time horizon T. In this work we focus on the discrete time model of the problem and show how it can be solved by the idea of time expanded network. Therefore after introducing both continuous and discrete time model, the method of the time expanded network will be discussed as the essential approach. One can see that the continuous time model of the problem is solvable by partitioning time horizon T to time steps $t = 0, 1, ..., T-1$, approximately.

3.1. *The Continuous Time Model*

The MCDF problem on a DMGNF is the problem of sending flow of several different commodities from multi generative nodes to multi consumer nodes at minimum total cost such that the flow going through arcs does not exceed their capacities. Therefore, the problem consists in finding a feasible dynamic flow, satisfying (1)-(4) that minimizes the following objective function:

$$\sum_{q\in \mathrm{K}}\sum_{(i,j)\in \mathrm{A}}\int_0^T c_{ijq}(t)x_{ijq}(t)dt \tag{5}$$

Where $c_{ijq}:[0,T]\rightarrow R^{+}$.
Therefore, in the continuous time model the MCDF problem may be formulated as follows:

$$\min\sum_{q\in \mathrm{K}}\sum_{(i,j)\in \mathrm{A}}\int_0^T c_{ijq}(t)x_{ijq}(t)dt$$

Subject to (1)-(4)

In order to a feasible flow exist during the time horizon T we require that:

$$\sum_{i\in V}\int_0^{\theta} v_{iq}(t)dt = 0\,\forall\theta\in[0,T],\forall q\in \mathrm{K} \tag{6}$$

It is easy to show that this condition is necessary but it is not sufficient one. In order to obey the time horizon T, we require that $x_{ijq}(t)=0$,$\forall q\in \mathrm{K}$,$\forall t>T$. To simplify our notation, we may sometimes use $x_{ijq}(t)=0$,$\forall q\in \mathrm{K}$, $\forall t\notin[0,T]$.

3.2. *The Discrete Time Model*

A dynamic multi generative network $N=(V,\mathrm{A},k,T,u,c)$ with node set V , $|V|=n$, arc set A, $|\mathrm{A}|=m$, capacity functions $u:\mathrm{A}\times\mathrm{K}\times\mathrm{N}\rightarrow R^{+}$ and $u(T):\mathrm{A}\rightarrow R^{+}$ and cost function $c:\mathrm{A}\times\mathrm{K}\times\mathrm{N}\rightarrow R^{+}$, where $\mathrm{N}=\{0,1,...,T-1\}$ and $\mathrm{K}=\{1,2,...,k\}$, is considered. If one unit of flow of commodity q leaves node i at time t on arc(i,j), one unit of flow of commodity q arrives at node j at the same time. The time horizon T is the time until which the flow can travel in the network and it defines set $\mathrm{N}=\{0,1,...,T-1\}$ consist in time steps.
In order to a feasible flow exist during the time horizon T we require that

$$\sum_{i\in V}\sum_{t=0}^{\theta} v_{iq}(t)=0\forall\theta\in\mathrm{N},\forall q\in\mathrm{K} \tag{7}$$

It is evident that this condition is necessary but it is not sufficient one. As before, every node $i\in V$ can serve as a multi generative node, a multi consumer node or a transshipment node.

The feasibility constraints for a flow in this case are the same as continuous case. It is sufficient to replace in (1)-(4), $\int_0^{\theta}$ with $\sum_0^{\theta}$ and $\forall\theta\in[0,T]$ with $\forall\theta\in\mathrm{N}$.
In this case $x_{ijq}(t)$ is the amount of flow of product q passing arc (i,j) at time step $t\in[0,T]$. It is easy to observe that the flow of any product does not enter arc (i,j) at time step t if $t\geqslant T$.
It can be seen that the conservation constraints in the discrete time model are equivalent to

$$\sum_j x_{ijq}(t)-\sum_j x_{jiq}(t)=v_{iq}(t)\forall i\in V,\forall q\in\mathrm{K},\forall t\in\mathrm{N} \tag{8}$$

Since $\sum_j\sum_{t=0}^{\theta}x_{ijq}(t)-\sum_j\sum_{t=0}^{\theta}x_{ijq}(t)=\sum_{t=0}^{\theta}(\sum_j x_{ijq}(t)-\sum_j x_{jiq}(t))$.
The total cost of discrete dynamic flow x at time horizon T in N may be defined as follows:

$$\sum_{q\in\mathrm{K}}\sum_{t\in\mathrm{N}}\sum_{(i,j)\in\mathrm{A}}c_{ijq}(t)x_{ijq}(t) \tag{9}$$

The MCDF problem on multi generative network with discrete time consists in finding a feasible discrete dynamic flow that minimizes objective function (9).
One can observe that if $v_{iq}(t)=v_{iq}$ where $v_{iq}\in R$, $\forall i\in V$, $\forall q\in\mathrm{K}$ and $T=0$ then the formulated problem becomes a static minimum cost multi-commodity flow problem on a static network.

3.3. *Feasibility Conditions*

Before stating the feasibility conditions of the problem, we need to define cut-sets: If $\varphi\neq S\subset V$ and $\overline{S}=V-S$ then cut-set $(S,\overline{S})$ is defined as $(S,\overline{S})=\{(i,j)\in\mathrm{A}:i\in S,j\in\overline{S}\}$.
In the static network flows the necessary and sufficient conditions for the feasibility are as following: For every set of nodes $S\subset V$

$$\sum_{i\in S}b(i)\leqslant\sum_{(i,j)\in(S,\overline{S})}u_{ij}$$
$$\sum_{i\in V}b(i)=0$$

Where $b(i)$ is the constant supply/demand of node i.
These conditions may be generalized for the DMGNF as a Theorem:

Theorem 3.1 The necessary and sufficient conditions for a multi generative network flow to be feasible are: for every cut-set $(S,\overline{S})$

$$\sum_{i\in S}\sum_{t=0}^{\theta} v_{iq}(t) \leqslant \sum_{(i,j)\in(S,\overline{S})}\sum_{t=0}^{\theta} u_{ijq}(t) \forall\theta\in \mathrm{N}, \forall q\in \mathrm{K} \tag{10}$$

$$\sum_{i\in S} V_i(T) \leqslant \sum_{(i,j)\in(S,\overline{S})} \mathbf{u}_{ij}(T) \tag{11}$$

$$\sum_{i\in V}\sum_{t=0}^{\theta} v_{iq}(t) = 0 \forall\theta\in \mathrm{N}, \forall q\in \mathrm{K} \tag{12}$$

Proof Suppose that $N=(V,\mathrm{A},k,T,u)$ is a dynamic multi generative network flow. Let $t\in \mathrm{N}$ be an arbitrary time step and $q\in \mathrm{K}$. The necessary and sufficient conditions for the feasibility of the MCDF problem on$N$ at time step t with respect to commodity$q$ are: For any $\varphi\neq S\subset V$ the following conditions hold ([1, 6]):

$$\sum_{i\in S} v_{iq}(t) \leqslant \sum_{(i,j)\in(S,\overline{S})} u_{ijq}(t),$$
$$\sum_{i\in V} v_{iq}(t) = 0.$$

Let $\theta\in \mathrm{N}$ be an arbitrary time point in$\mathrm{N}$. Then by summing the above inequalities up toθ , we get the following conditions:

$$\sum_{t=0}^{\theta}\sum_{i\in S} v_{iq}(t) = \sum_{i\in S}\sum_{t=0}^{\theta} v_{iq}(t)$$
$$\leqslant \sum_{t=0}^{\theta}\sum_{(i,j)\in(S,\bar{S})} u_{ijq}(t)$$
$$= \sum_{(i,j)\in(S,\overline{S})}\sum_{t=0}^{\theta} u_{ijq}(t),$$
$$\sum_{t=0}^{\theta}\sum_{i\in V} v_{iq}(t) = \sum_{i\in V}\sum_{t=0}^{\theta} v_{iq}(t) = 0.$$

Note that $S\subset V$ is an arbitrary subset of V at any step. Also for feasibility of the problem at T, it is necessary and sufficient that the following conditions hold:

$$\sum_{i\in S} V_i(T) \leqslant \sum_{(i,j)\in(S,\overline{S})} \mathbf{u}_{ij}(T) \,\forall \text{ cut-set } (S,\overline{S}).$$

Note that a similar Theorem is true for the continuous case and conditions:

$$\sum_{i\in S}\int_0^{\theta} v_{iq}(t)dt \leqslant \sum_{(i,j)\in(S,\overline{S})}\int_0^{\theta} u_{ijq}(t)dt \forall\theta\in[0,T], \forall q\in \mathrm{K}$$
$$\sum_{i\in S} V_i(T) \leqslant \sum_{(i,j)\in(S,\overline{S})} \mathbf{u}_{ij}(T),$$
$$\sum_{i\in V}\int_0^{\theta} v_{iq}(t)dt = 0 \forall\theta\in[0,T], \forall q\in \mathrm{K}$$

4. The Method for Solving the MCDF Problem

To solve the formulated problem, we propose an approach based on the reduction of the minimum cost dynamic flow problem on multi generative network to a minimum cost static flow problem on a static network. We consider the discrete time model, in which all times are integral and bounded by horizon T. We show that the dynamic problem on multi generative network N can be reduced to static one on an auxiliary network N_{K}^{T} , which we call time-commodity expanded network. The most important advantage of this method is that it turns the problem of determining an optimal dynamic flow on a multi generative network into a classical static network flow problem. The essence of the time-commodity expanded network is that it contains a copy of the node set and the arc set of the multi generative network for each $t\in \mathrm{N}$ and $q\in \mathrm{K}$. We define N_{K}^{T} as:

1. $V_{\mathrm{K}}^{T} = \{i_q^t \mid t\in \mathrm{N}, q\in \mathrm{K}, i\in V\}$
2. $\mathrm{A}_{\mathrm{K}}^{T} = \{(i,j)_q^t = (i_q^t, j_q^t) \mid t\in \mathrm{N}, q\in \mathrm{K}, (i,j)\in \mathrm{A}\}$
3. $v_{iq}^t = v_{iq}(t)$ for $i_q^t\in V_{\mathrm{K}}^{T}$
4. $u_{ijq}^t = u_{ijq}(t)$ for $(i,j)_q^t\in \mathrm{A}_{\mathrm{K}}^{T}$
5. $c_{ijq}^t = c_{ijq}(t)$ for $(i,j)_q^t\in \mathrm{A}_{\mathrm{K}}^{T}$
6. $u_{ij}^{T} = u_{ij}(T)$ for $\{(i,j)_q^t \mid t\in \mathrm{N}, q\in \mathrm{K}\}$

Note that, for the networkN_{K}^{T} , $\left|V_{\mathrm{K}}^{T}\right| = nkT$ and $\left|\mathrm{A}_{\mathrm{K}}^{T}\right| = mkT$. In the time-commodity expanded network, we can consider the problem as a static network flow problem. It will be illustrated that every static flow x_q^t in N_{K}^{T} corresponds to a discrete dynamic flow x with time horizon T in N and vice-versa. In the following we state a correspondence between dynamic flows in N and static flows in the static time-commodity expanded networkN_{K}^{T} . For this aim, let $x(t)$ be a flow in$N$, then

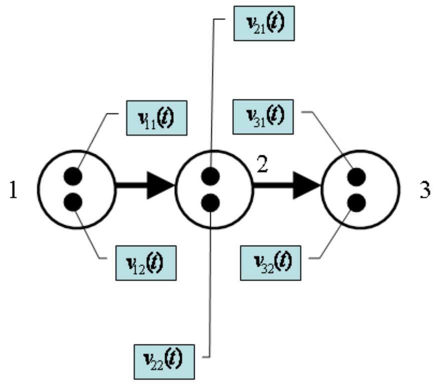

Fig. 1. The dynamic multi generative dynamic network N.

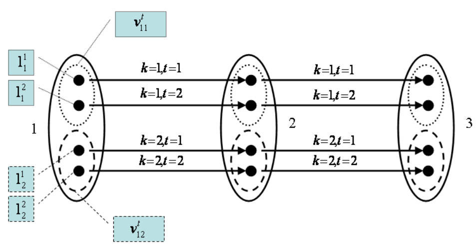

Fig. 2. The time-commodity expanded network N^T_{K}.

the following function

$$x^t_q : \mathrm{A}^T_K \to R^+ \, such that \, x^t_{ijq} = x_{ijq}(t) \tag{13}$$

represents a flow in the time-commodity expanded networkN^T_{K} where x^t_{ijq} denotes the amount of flow passing through the arc$(i,j)^t_q \in \mathrm{A}^T_{\mathrm{K}}$.
Let us construct the time-commodity expanded network N^T_{K} for N given in Figure 1. The set of time steps is $\mathrm{N} = \{1,2\}$ and the set of products of generative functions is$\mathrm{K} = \{1,2\}$. So each node i consists in two time functions $v_{i1}(t)$ and$v_{i2}(t)$. The time functions $p_{iq}(t)$ and$r_{iq}(t)$, time-commodity dependent capacities$u_{ijq}(t)$, horizon capacities $u_{ij}(T)$ and time-commodity varying costs $c_{ijq}(t)$ are considered to be given.

Now, in accordance with the definition of time-commodity expanded network and our notions in section 3.2, we get the following equivalent formulation of the MCDF problem on the static time-commodity expanded networkN^T_{K}:

$$\min \sum_{q\in \mathrm{K}} \sum_{t \in \mathrm{N}} \sum_{(i,j)^t_q \in \mathrm{A}^T_{\mathrm{K}}} c^t_{ijq} x^t_{ijq} \tag{14}$$

$$\sum_j x^t_{ijq} - \sum_j x^t_{jiq} = v^t_{iq} \forall i^t_q \in V^T_{\mathrm{K}}, q \in \mathrm{K}, t \in \mathrm{N} \tag{15}$$

$$0 \leqslant x^t_{ijq} \leqslant u^t_{ijq} \forall (i,j)^t_q \in \mathrm{A}^T_{\mathrm{K}}, q \in \mathrm{K}, t \in \mathrm{N} \tag{16}$$

$$\sum_{q\in \mathrm{K}} \sum_{t \in \mathrm{N}} x^t_{ijq} \leqslant u^T_{ij} \{(i,j)^t_q \mid t \in \mathrm{N}, q \in \mathrm{K}\} \tag{17}$$

Note that, constraint (17) corresponds to a subset of arcs in the time-commodity expanded network instead of a constraint on each arc separately. In fact, this constraint guaranties the total flow traversing the arc throughout the time horizon T is at most $u^T_{ij}\ [= u_{ij}(T)]$. We refer to this constraint as collective constraint.
By following lemma and theorem, we will show that the dynamic flow problem on multi generative network N may be solved as a static flow problem on the auxiliary network N^T_{K} . Similar theorems and lemmas can be found in the literature in the context of flows over time (for example, see [16]).

Lemma 4.1 Let $x^t_q : \mathrm{A}^T_{\mathrm{K}} \to R^+$ be a flow in the static networkN^T_{K} . Then the function $x : \mathrm{A} \times \mathrm{K} \times \mathrm{N} \to R^+$ defined in the following way:

$$x_{ijq}(t) = x^t_{ijq} for (i,j) \in \mathrm{A}, (i,j)^t_q \in \mathrm{A}^T_{\mathrm{K}}, q \in \mathrm{K}, t \in \mathrm{N}$$

represents a dynamic flow of a set of products in the multi generative network N. Let $x : \mathrm{A} \times \mathrm{K} \times \mathrm{N} \to R^+$ be a dynamic flow of a set of products in the multi generative network$N$. Then the function $x^t_q : \mathrm{A}^T_{\mathrm{K}} \to R^+$ defined in the following way:

$$x^t_{ijq} = x_{ijq}(t) \text{ for} (i,j)^t_q \in \mathrm{A}^T_{\mathrm{K}}, (i,j) \in \mathrm{A}, q \in \mathrm{K}, t \in \mathrm{N}$$

represents a static flow in the networkN^T_{K}.

Proof It is obvious that the foregoing correspondences are bijections from the set of T-horizon flows with a set of products in$N$ onto the set of flows in the time-commodity expanded network N^T_{K} and vice-versa. We have to show that each static flow in the time-commodity expanded network N^T_{K} is put into the correspondence with a dynamic flow of a set of products in the dynamic multi generative network N and vice-versa. To prove the first part of the lemma, we will show that conditions (1)-(4) for $x : \mathrm{A} \times \mathrm{K} \times \mathrm{N} \to R^+$ in N are true in the discrete time model. These conditions evidently result from the conditions (15)-(17) and definition (13) in the static networkN_{K}^T . To prove the second part, it is sufficient to show that conditions (15)-(17) hold for $x_q^t : \mathrm{A}_{\mathrm{K}}^T \to R^+$. Correctness of these conditions easily results from the procedure of constructing the time-commodity expanded network and the correspondence between flows in the static and multi dynamic networks.

Let $x_{ijq}(t)$ be a dynamic flow of a set of products in$N$, and let x_{ijq}^t be a corresponding function inN_{K}^T . Lets prove that x_{ijq}^t satisfies the conservation constraints inN_{K}^T . Let $i \in V$ be an arbitrary node in N and $t \in \mathrm{N}$, an arbitrary time step and$q \in \mathrm{K}$, an arbitrary product:

$$\begin{aligned} v_{iq}(t) &= \sum_j x_{ijq}(t) - \sum_j x_{jiq}(t) \\ &= \sum_j x_{ijq}^t - \sum_j x_{jiq}^t = v_{iq}^t \end{aligned}$$

According to definition of the time-commodity expanded network all necessary conditions are satisfied for each node $i_q^t \in V_{\mathrm{K}}^T$. Hence, x_{ijq}^t is a flow inN_{K}^T . It is easy to verify that a feasible flow in N is a feasible flow in the time-commodity expanded network N_{K}^T and vice-versa. Indeed,

$$\begin{aligned} &0 \leqslant x_{ijq}^t = x_{ijq}(t) \leqslant u_{ijq}(t) = u_{ijq}^t \\ &\sum_{q\in\mathrm{K}}\sum_{t\in\mathrm{N}} x_{ijq}^t = \sum_{q\in\mathrm{K}}\sum_{t\in\mathrm{N}} x_{ijq}(t) \leqslant u_{ij}(T) = u_{ij}^T . \square \end{aligned}$$

The total cost of the static flow x_q^t in N_{K}^T may be determined as follows:

$$c_q^t(x_q^t) = \sum_{q\in\mathrm{K}}\sum_{t\in\mathrm{N}}\sum_{(i,j)_q^t\in\mathrm{A}_{\mathrm{K}}^T} c_{ijq}^t x_{ijq}^t \qquad (18)$$

Theorem 4.1 If x is a flow in the dynamic multi generative network N and x_q^t is a corresponding flow in the time-commodity expanded network N_{K}^T, then $c_q^t(x_q^t) = c(x)$. Moreover, if $x_q^{t^*}$ is a minimum cost flow in N_{K}^T , then the corresponding dynamic flow x^* of a set of products in N is also a minimum cost one and vice-versa.

Proof Let $x : \mathrm{A} \times \mathrm{K} \times \mathrm{N} \to R^+$ be an arbitrary dynamic flow in N . According to Lemma 4.1 the unique flow x_q^t in N_{K}^T corresponds to flow x in N, and therefore we have:

$$\begin{aligned} c(x) &= \sum_{q\in\mathrm{K}}\sum_{t\in\mathrm{N}}\sum_{(i,j)\in\mathrm{A}} c_{ijq}(t) x_{ijq}(t) \\ &= \sum_{q\in\mathrm{K}}\sum_{t\in\mathrm{N}}\sum_{(i,j)_q^t\in\mathrm{A}_{\mathrm{K}}^T} c_{ijq}^t x_{ijq}^t \\ &= c_q^t(x_q^t). \end{aligned}$$

To prove the second part of the theorem we again use Lemma 4.1. Let $x^* : \mathrm{A} \times \mathrm{K} \times \mathrm{N} \to R^+$ be the optimal dynamic flow in N and $x_q^{t^*}$ be the corresponding optimal flow inN_{K}^T . Then

$$\begin{aligned} c(x^*) &= \sum_{q\in\mathrm{K}}\sum_{t\in\mathrm{N}}\sum_{(i,j)\in\mathrm{A}} c_{ijq}(t) {x_{ijq}}^*(t) \\ &= \sum_{q\in\mathrm{K}}\sum_{t\in\mathrm{N}}\sum_{(i,j)_q^t\in\mathrm{A}_{\mathrm{K}}^T} c_{ijq}^t {x_{ijq}^t}^* \\ &= c_q^t(x_q^{t^*}). \end{aligned}$$

The converse proposition is proved in an analogous way.□

According to definition of time-commodity expanded network, the problem of MCDF on N_{K}^T can be reduced to the following matrix form, (19), as a linear program whose special structure permits efficient computations of solution.

$$\begin{aligned} &\min \sum_{q\in\mathrm{K}}\sum_{t\in\mathrm{N}} \mathbf{c}_q^t \mathbf{x}_q^t \\ &\mathbf{A}\mathbf{x}_q^t = \mathbf{v}_q^t \forall q \in \mathrm{K}, \forall t \in \mathrm{N} \\ &0 \leqslant \mathbf{x}_q^t \leqslant \mathbf{u}_q^t \forall q \in \mathrm{K}, \forall t \in \mathrm{N} \\ &0 \leqslant \sum_{q\in\mathrm{K}}\sum_{t\in\mathrm{N}} \mathbf{x}_q^t \leqslant \mathbf{u}^T \end{aligned} \qquad (19)$$

Where $\mathbf{x}_q^t = \left\{x_{ijq}^t\right\}_{i,j}$ is the vector of flows of commodity$q \in \mathrm{K}$ respect to t in N_{K}^T. $\mathbf{u}_q^t = \left\{\mathbf{u}_{ijq}^t\right\}_{i,j}$ represents the vector of upper limits on flows of commodity$q \in \mathrm{K}$ respect to t in N_{K}^T. Also $\mathbf{u}^T = \left\{u_{ij}^T\right\}_{i,j}$ represents the vector of upper limits on the total flow of total commodities flowing in the arcs for all $t \in \mathrm{N}$ in N_{K}^T and $\mathbf{c}_q^t = \left\{c_{ijq}^t\right\}_{i,j}$ represents the vector of arc costs for commodity q respect to t in N_{K}^T. A is the node-arc incidence matrix of the graph. Finally, $\mathbf{v}_q^t = \left\{v_{iq}^t\right\}_i$ represent the vector of demands for commodity q respect to t (i.e., generative demands, at the time step t for commodity q) in N_{K}^T. It can be seen that we can formulate the MCDF problem as the minimum cost static flow network problem that possesses the block diagonal structure. Thus we may apply the block diagonal decomposition technique and some other methods to the foregoing problem ([1, 2, 3, 15, 21]).

Note that the foregoing formulation of the MCDF problem has a simple constraint structure. It contains $m+nkT$ constraints since it contains one mass balance constraint for every node at every step for each product and one horizon capacity constraint for every arc, and has $m+mkT$ variables (including the slack variable). Let $\mathrm{X}_q^t = \left\{\mathbf{x}_q^t : \mathbf{A}\mathbf{x}_q^t = \mathbf{v}_q^t, 0 \leqslant \mathbf{x}_q^t \leqslant \mathbf{u}_q^t\right\}$. Let us assume that each component of $\mathbf{u}_q^t$ is finite so that X_q^t for every t, $t \in \mathrm{N}$ and q , $q \in \mathrm{K}$ is bounded. Then $\mathbf{x}_q^t$ can be expressed as a convex combination of the extreme points of X_q^t as follows ([3]):

$$\mathbf{x}_q^t = \sum_{i=1}^{k_q^t} \lambda_{iq}^t \mathbf{x}_{iq}^t$$

Where

$$\sum_{i=1}^{k_q^t} \lambda_{iq}^t = 1,$$
$$\lambda_{iq}^t \geqslant 0 i = 1, ..., k_q^t, q \in \mathrm{K}, t \in \mathrm{N}$$

and $\mathbf{x}_{1q}^t, \mathbf{x}_{2q}^t, ..., \mathbf{x}_{k_q^t q}^t$ are the extreme points of X_q^t. Substituting for $\mathbf{x}_q^t$ in the MCDF problem and denoting the vector of slack variables by$\mathbf{s}$, we get the following formulation (20):

$$\min \sum_{q\in\mathrm{K}} \sum_{t\in\mathrm{N}} \sum_{i=1}^{k_q^t} (\mathbf{c}_q^t \mathbf{x}_{iq}^t) \lambda_{iq}^t$$
$$\sum_{q\in\mathrm{K}} \sum_{t\in\mathrm{N}} \sum_{i=1}^{k_q^t} (\mathrm{I}\mathbf{x}_{iq}^t) \lambda_{iq}^t + \mathbf{s} = \mathbf{u}^T$$
$$\sum_{i=1}^{k_q^t} \lambda_{iq}^t = 1 \forall q \in \mathrm{K}, \forall t \in \mathrm{N}$$
$$\lambda_{iq}^t \geqslant 0 \forall q \in \mathrm{K}, \forall t \in \mathrm{N}, i = 1, ..., k_q^t$$
$$\mathbf{s} \geqslant 0 \tag{20}$$

Lemma 4.2 In the above formulation of the MCDF problem, any linear programming basis for the MCDF problem will contain $m + kT$ basic variables in λ_{iq}^t space, and will contain at least one λ_{iq}^t with respect to each $t \in \mathrm{N}$ and commodity, $q \in \mathrm{K}$. Moreover, any basis detects at least one arc set at every time step $t \in \mathrm{N}$ for each commodity $q \in \mathrm{K}$ transports a positive flow.

Proof Indeed, the foregoing formulation of MCDF problem contains $m + kT$ constraints. Therefore, any linear programming basis will contain $m + kT$ basic variables in λ_{iq}^t space and it is found that any basis will contain at least one λ_{iq}^t with respect to each t and commodity q ([3,15]). Hence, any basis detects at least one arc set with respect to each $t \in \mathrm{N}$ and commodity$q \in \mathrm{K}$ which transports a positive flow. That means, any basis detects one arc set at every time step t for every commodity q that it transports a positive flow.□

5. The Path Formulation of the MCDF Problem

To simplify our discussion, we consider a special case of the MCDF problem: we assume that in the main problem there is a single source s and a single sink d for all products and a time-commodity flow requirement $r(t, q)$ between this source and sink node. So, we consider a single source s_q^t and a single sink d_q^t for each$q \in \mathrm{K}$ and $t \in \mathrm{N}$ in N_{K}^T and a flow requirement of r_q^t units between these sources and sinks. We also assume that there is no time-commodity dependent capacity $u_{ijq}(t)$. Let us first reformulate the MCDF problem on N_{K}^T using path and cycle flows instead of arc flows. Therefore, we assume that we can represent any potentially optimal solution as sum of flows on directed paths ([1]). For each product q respect to each t , Let P_q^t denote the collection of all directed paths from the source node s_q^t to the sink node d_q^t in the static network N_{K}^T (i.e., P_q^t denote the collection of all directed paths from the source node s to the sink node d in the dynamic network N at time moment t for commodity q). In the path flow formulation, each decision variable $f(p)$ is the flow on some path p and for the step t and product q. We define this variable for every directed path p in P_q^t. Let $\delta_{ijq}^t(p)$ be an arc-flow indicator variable, that is, $\delta_{ijq}^t(p)$ equals 1 if arc $(i, j)_q^t$ is contained in the path p, and is 0 otherwise. Note that if $(i, j)_\beta^\alpha \in p$ and $p \in \mathrm{P}_q^t$ then $\alpha = t$ and $\beta = q$.

The flow decomposition theorem of network flows ([1]) states that we can always decompose optimal flow x_{ijq}^t into path flows $f(p)$ as $f(P) = \sum_{(i,j)_q^t \in A_K^T} \delta_{ijq}^t(p) f(p)$. Let $c_q^t(p) = \sum_{(i,j)_q^t \in \mathrm{A}_{\mathrm{K}}^T} c_{ijq}^t \delta_{ijq}^t(p) = \sum_{(i,j)_q^t \in P_q^t} c_{ijq}^t$ denote the per unit cost of flow on the path $p \in \mathrm{P}_q^t$ with respect to the $t \in \mathrm{N}$ and product$q \in \mathrm{K}$. Note that for each t and q, if we substitute for the arc flow variable in

the objective function, we find that $\sum_{(i,j)_q^t \in \mathrm{A}_\mathrm{K}^T} c_{ijq}^t x_{ijq}^t = \sum_{(i,j)_q^t \in \mathrm{A}_\mathrm{K}^T} c_{ijq}^t \sum_{P \in \mathrm{P}_q^t} \delta_{ijq}^t(p) f(p) = \sum_{P \in \mathrm{P}_q^t} c_q^t(p) f(p)$.
This observation shows that we can express the cost of any solution as either the cost of arc flows or the cost of path flows. Then we obtain the following equivalent path flow formulation of the MCDF problem:

$$\min \sum_{q \in \mathrm{K}} \sum_{t \in \mathrm{N}} \sum_{P \in \mathrm{P}_q^t} c_q^t(p) f(p) \tag{21}$$

$$\sum_{q \in \mathrm{K}} \sum_{t \in \mathrm{N}} \sum_{p \in \mathrm{P}_q^t} \delta_{ijq}^t(p) f(p) \leqslant u_{ij}^T \forall (i,j) \in \mathrm{A} \tag{22}$$

$$\sum_{P \in \mathrm{P}_q^t} f(p) = r_q^t \forall q \in \mathrm{K}, \forall t \in \mathrm{N} \tag{23}$$

$$f(p) \geqslant 0 \forall p \in \mathrm{P}_q^t, q \in \mathrm{K}, t \in \mathrm{N} \tag{24}$$

In this formulation by using the flow decomposition any feasible arc flow of the system $\mathbf{A}\mathbf{x}_q^t = \mathbf{v}_q^t$ in (19) has decomposed into a set of path flows in such a way that path flows satisfy the mass balance condition (23). This formulation of the MCDF has a single constraint (22) for each arc (i,j) which state that the total path flows of total products passing through the arc at all time steps is at most u_{ij}^T , the horizon capacity of the arc (i.e., the total path flows of total products passing through the arc during the time horizon T is at most u_{ij}^T). Moreover, the problem has a single constraint (23) with respect to each $t \in \mathrm{N}$ and $q \in \mathrm{K}$, which states that the total of flow on all paths connecting the source nodes s_q^t and sink nodes d_q^t with respect to t for product q must equal the demandr_q^t. (i.e., the total flow of commodity q on all paths connecting the source node s and sink node d in the dynamic network N at moment t must equal the demand of this commodity q at this moment of time t).
For a network with n nodes, m arcs, k types of time functions and time horizon T, the path formulation of the MCDF problem contains $m + kT$ constraints. In contrast, the arc formulation stated in section 4 contains $m + nkT$ constraints. We can apply the generalized upper bounding simplex method, to solve the new formulation very efficiently ([1, 21]).
For example, a network with $n = 1000$ nodes, $m = 5000$ arcs, $k = 10$ types of time functions and with a time-step set $\mathrm{N}(|\mathrm{N}| = 1,000,000)$, the path flow formulation contains $10,005,000$ constraints. In contrast, the arc formulation (matrix form) contains $10,000,005,000$ constraints. But on other hand, because no path appears in more than one of the constraint (23), we can apply a generalized version of the simplex method, known as generalized upper bounding simplex method (GUBS), to solve the path flow formulation very efficiently. Even though the linear programming basis for our example has size $10,005,000 \times 10,005,000$, the GUBS method is able to perform all of its matrix computations on a very much smaller basis of size 5000×5000. This method essentially solves the problem as though it contained only m master constraints (in our formulation, that is constraint (22)), which, for this sample data, means that we can essentially solve a linear program with only 5000 constraints instead of over 10 billion constraints in the arc formulation.

Lemma 5.1 In the path formulation of the MCDF problem, any linear programming basis contains $m + kT$ basic variables. Moreover, any basis detects at least one path at every time step $t \in \mathrm{N}$ for every commodity $q \in \mathrm{K}$ which transports a positive flow.
Proof As mentioned before, the path formulation of MCDF problem contains $m + kT$ constraints, and consequently any linear programming basis will contain $m + kT$ basic variables. And it is found that ([1, 3, 21]) in any basis, at least one path with respect to each t and q must carry a positive flow. That means, any basis will detect at least one path at every moment of time t for each commodity q that it transports a positive flow. □

6. Conclusion

In this paper, we investigated and solved the problem of minimum cost dynamic flow on dynamic multi generative networks in which we assigned to each node$k$ time functions. As you saw, in these networks supplies and demands were not given as fixed values; instead, they were continuously generated. Therefore, in such kind of network flows, the amount of flow (of any product) on each arc changed at every moment of time during a pre-specified time horizon. Moreover, by transforming the multi generative network to a static network, called time-commodity expanded network, we showed that the dynamic problem can be solved as a static minimum cost flow problem with a special structure and efficient algorithms for finding optimal flows were proposed.

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